Conditions for the Existence and Stability of the Continuous Attractor in the Classical XY Model with an Associative-Memory-Type Interaction

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We analyze the structure of attractors in the classical XY model with an associative-memory-type interaction by the statistical mechanical method. Previously, it was found that when patterns are uncorrelated, points on a path connecting two memory patterns in the space of the order parameters are solutions of the saddle point equations (SPEs) in the case that p is $\mathcal{O}(1)$ irrespective of N and $N \gg 1$, where p and N are the numbers of patterns and spins, respectively. This state is called the continuous attractor (CA). In this paper, we clarify the conditions for the existence and stability of the CA with and without the correlation a $(0 \leq a < 1)$ between any two patterns in the case that $N \gg 1$ and the self-averaging property holds. We find that the CA exists for any $p \geq 2$ when $a = 0$, but it exists only for $p = 2$ when $0 < a < 1$ and for $p = 3$ when $a < 1/3$. For $p = 2$ and 3, and for $a < 1$, we analyze the SPEs and find all solutions and study their stabilities. We perform Markov chain Monte Carlo simulations and compare numerical and theoretical results. We find that for a finite system of size N and for $a = 0$, owing to the breakdown of the self-averaging property, the CA ceases to exist at a finite value of p. We define the critical value of p_c until which the CA exists and numerically study the system size N dependence of p_c. We find that the numerical results are consistent with the theoretical results obtained by taking into account the breakdown of the self-averaging property. Furthermore, for $a > 0$, we numerically study the case that patterns are subject to external noise and find that p_c increases as the noise amplitude increases.

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1. Introduction

Since Hopfield proposed a model of the associative memory of a neural network,[1] many studies on the subject have been carried out from the viewpoint of statistical mechanics[2)–7)]. In many studies, states of neurons are represented by Ising spins as in the Hopfield model. In our previous study,[8] however, we adopted the classical XY spins as states of neurons. The main motivation for this is that we wanted to construct an associative memory model with the following properties that real brains have. In real brains, different memories spontaneously appear one after another, and by an external stimulus, a memory related to the stimulus is retrieved. That is, it seems that many memories in a real brain are "connected" in a sense. We expected that associative memory models composed of the XY spins would have such connected memories because they have a continuous degree of freedom, contrary to models composed of the Ising spins, which have only isolated memories, i.e., point attractors.
We analyzed the XY spin system with the associative memory interaction by the statistical mechanical method in the case that p is $\mathcal{O}(1)$ irrespective of N and $N \gg 1$, where p and N are the numbers of patterns and spins, respectively, when patterns are uncorrelated. We derived the saddle point equations (SPEs) for the order parameters, and by numerically solving the SPEs we found a new type of attractor, the so-called continuous attractor (CA). The CA is a one-parameter family of solutions of the SPEs, and the points on a path connecting any two memory patterns in the space of order parameters become solutions, which we expected to exist in the XY spin system. See Fig. 1. We performed Markov chain Monte Carlo simulations (MCMCs) and confirmed the theoretical results numerically.

In this paper, we study the two cases that patterns are uncorrelated and correlated, in the case that $N \gg 1$ and the self-averaging property holds. Let a be the correlation between any two patterns, $0 \le a < 1$. By introducing sublattices, we rewrite the SPEs in a compact form, which allows us to characterize the CA and enables us to study solutions of the SPEs and their stabilities analytically. Then, we find the conditions for the existence with and without the correlation a. The CA exists for any p when $a = 0$, whereas it exists only for $p = 2$ when $0 \le a < 1$ and for $p = 3$ when $a < \frac{1}{3}$. We perform MCMCs and compare numerical and theoretical results. When $a = 0$, contrary to the theoretical result, numerical results show that the CA ceases to exist at a finite value of p. We define the critical value of p_c until which the CA exists

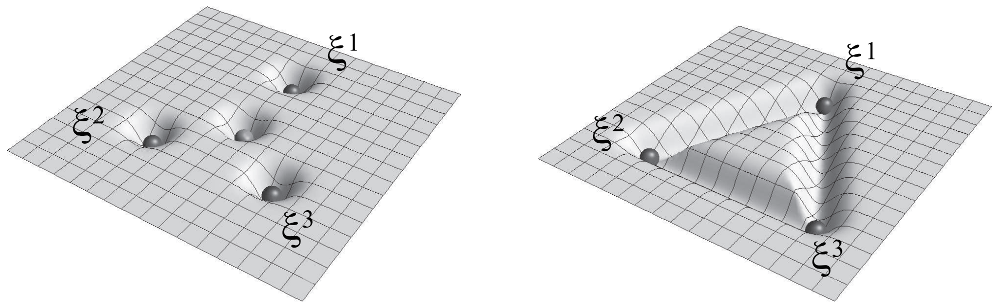

Fig. 1. Schematic figures of point attractors and continuous attractors. ξ denotes a pattern. Left: dips represent point attractors. The dip in the middle is a mixed state composed of three patterns. Right: valleys represent continuous attractors.

and numerically study the N dependence of p_c. We find that the numerical results are consistent with the theoretical results obtained by taking into account the breakdown of the self-averaging property. For $a > 0$, we confirm the theoretical results by numerical simulations. Furthermore, for $a > 0$, we numerically study the case that patterns are subject to external noise and find that p_c increases as the noise amplitude increases.

The structure of this paper is as follows. In sect. 2, we analyze the SPEs, rewrite them by introducing sublattices, and show the list of stable solutions for $p \leq 3$. In sect. 3, we characterize the CA and derive the conditions for its existence. In sect. 4, we study the stabilities of the relevant solutions, mainly for $p \leq 3$, by calculating the Hessian matrix. In sect. 5, we show numerical results for the phase diagram in the (a, T) plane, the temperature dependences of order parameters, the N dependence of p_c, and the effects of noise input to patterns. Section 6 contains a summary and discussion of the results. In Appendix A, we derive the expressions for the free energy and the SPEs, and we describe the properties of the function $u(x)$ that appears in the SPEs in Appendix B. In Appendix C, we give proofs of relations among variables related to sublattices. We derive all solutions of the SPEs for $p \leq 3$ in Appendix D. The stabilities of irrelevant solutions of the SPEs for $p \leq 3$ are analyzed in Appendix E. In Appendix F, for $p = 3$, we derive the range of an order parameter that characterizes the CA, and relations between order parameters for the CA.

2. Analysis of the Saddle Point Equations

We study the XY model, which consists of N XY spins $\boldsymbol{X}_i = (\cos\phi_i, \sin\phi_i)$, $1 \leq i \leq N$, where ϕ_i is the phase of the ith XY spin. The Hamiltonian H for the XY model is given by

$$H = -\sum_{i<j} J_{ij} \cos(\phi_i - \phi_j). \tag{1}$$

The associative memory interaction is expressed as

$$J_{ij} = \frac{J}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu. \tag{2}$$

We assume that the μth memory pattern ξ_i^μ takes values of ± 1 and that a correlation exists between the memory patterns, which is represented by $\langle \xi_i^\mu \xi_j^\nu \rangle = a\delta_{ij}$ for $\mu \neq \nu$ and $\langle \xi_i^\mu \xi_j^\mu \rangle = \delta_{ij}$, where $\langle \cdots \rangle$ denotes the average over $\{\xi_i^\mu\}$. We assume $0 \leq a < 1$. The order parameter is defined by

$$R_{\mu R} = \frac{1}{N}\sum_{i=1}^{N} \xi_i^\mu \cos\phi_i, \tag{3}$$

$$R_{\mu I} = \frac{1}{N}\sum_{i=1}^{N} \xi_i^\mu \sin\phi_i. \tag{4}$$

The Hamiltonian is rewritten as follows:

$$H = -\frac{JN}{2}\sum_{\mu=1}^{p} R_\mu^2 + \frac{Jp}{2}, \tag{5}$$

$$R_\mu = \sqrt{R_{\mu R}^2 + R_{\mu I}^2}. \tag{6}$$

2.1 Free energy and saddle point equations

As is derived in Appendix A for $N \gg 1$, the free energy $F = -\frac{1}{\beta}\ln Z$ is expressed as Eq. (7), where $\beta = \frac{1}{T}$ and the Boltzmann constant is set to 1, $k_B = 1$,

$$F = \frac{JN}{2}R^2 - \frac{1}{\beta}\sum_{j=1}^{N} \ln(2\pi I_0(\beta J \Xi_j)), \tag{7}$$

where

$$R = \sqrt{\sum_{\mu=1}^{p} R_\mu^2}, \tag{8}$$

$$\Xi_j = \sqrt{(\sum_{\mu=1}^{p} \xi_j^\mu R_{\mu R})^2 + (\sum_{\mu=1}^{p} \xi_j^\mu R_{\mu I})^2}, \tag{9}$$

$$I_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{x\cos\phi}\cos(n\phi)d\phi. \quad (10)$$

$I_n(x)$ is the modified Bessel function of the first kind. The SPEs are obtained as

$$R_{\mu R} = \beta J \frac{1}{N}\sum_{j=1}^{N}\sum_{\nu=1}^{p} u(x_j)\xi_j^\mu \xi_j^\nu R_{\nu R}, \quad (11)$$

$$R_{\mu I} = \beta J \frac{1}{N}\sum_{j=1}^{N}\sum_{\nu=1}^{p} u(x_j)\xi_j^\mu \xi_j^\nu R_{\nu I}, \quad (12)$$

$$x_j = \beta J \Xi_j,\ u(x) = \frac{I_1(x)}{xI_0(x)}. \quad (13)$$

The function $u(x)$ has the following properties:

$$u(0) = \frac{1}{2},\ \lim_{x\to\infty} u(x) = 0,$$
$$u(x) > 0,\ \text{for } x \geq 0,\ \ u'(x) < 0,\ \text{for } x > 0.$$

See Appendix B for details. Figure 2 shows the graph of $u(x)$.

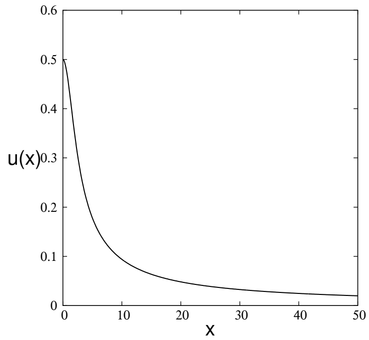

Fig. 2. Function $u(x)$.

We consider the case that the self-averaging property holds. That is,

$$\frac{1}{N}\sum_{j=1}^{N} g(\xi_j^\mu) = [g(\xi^\mu)], \quad (14)$$

where $[\cdot]$ means the average over $\{\xi_i^\mu\}$. Thus, we obtain

$$R_{\mu R} = \beta J \sum_{\nu=1}^{p} c_{\mu\nu} R_{\nu R}, \quad (15)$$

$$R_{\mu I} = \beta J \sum_{\nu=1}^{p} c_{\mu\nu} R_{\nu RI}. \quad (16)$$

Here, we define

$$c_{\mu\nu} = \left[u(x_j)\xi_j^\mu \xi_j^\nu\right] = c_{\nu\mu}. \quad (17)$$

Now, let us study whether or not reflection symmetry exists in the solutions of the SPEs. (11) and (12). Suppose that the order parameters $(\{R_{\nu R}\},\{R_{\nu I}\})$ are the solutions of the SPEs. Let us consider the order parameters in which the signs of $R_{\mu_0 R}$ and $R_{\mu_0 I}$ are reversed, that is, we consider $(R_{1R},\cdots,-R_{\mu_0 R},\cdots,R_{pR},R_{1I},\cdots,-R_{\mu_0 I},\cdots,R_{pI})$. We define $R'_{\mu_0 R}=-R_{\mu_0 R}$, $\xi'^{\mu_0}_j=-\xi^{\mu_0}_j$, and for $\mu\neq\mu_0$, $R'_{\mu R}=R_{\mu R}$ and $\xi'^{\mu}_j=\xi^{\mu}_j$. x_j is expressed as

$$x_j \quad = \quad \beta J\sqrt{(\sum_{\mu=1}^{p}\xi'^{\mu}_j R'_{\mu R})^2+(\sum_{\mu=1}^{p}\xi'^{\mu}_j R'_{\mu I})^2}. \tag{18}$$

Then, we find that $(\{R'_{\mu R}\},\{R'_{\mu I}\})$ satisfy the SPEs. (11) and (12) with ξ replaced by ξ'. Let ξ^0_j be the "mother" pattern, which takes values of ±1 with probability $\frac{1}{2}$ and produces $\xi^1_j,\cdots,\xi^p_j$. The conditional probability $P(\xi^\mu_j|\xi^0_j)$ of ξ^μ_j given ξ^0_j is

$$P(\xi^\mu_j|\xi^0_j) \quad = \quad \frac{1+\sqrt{a}}{2}\delta_{\xi^\mu_j,\xi^0_j}+\frac{1-\sqrt{a}}{2}\delta_{\xi^\mu_j,-\xi^0_j}. \tag{19}$$

Then, we obtain for $\mu\neq\nu$

$$\begin{aligned} P(\xi^\mu_j,\xi^\nu_j) \quad &= \quad \sum_{\xi^0_j}P(\xi^\mu_j|\xi^0_j)P(\xi^\nu_j|\xi^0_j)P(\xi^0_j) \\ &= \quad \frac{1+a}{4}(\delta_{\xi^\mu_j+\xi^\nu_j,2}+\delta_{\xi^\mu_j+\xi^\nu_j,-2})+\frac{1-a}{4}\delta_{\xi^\mu_j+\xi^\nu_j,0}. \end{aligned} \tag{20}$$

On the other hand, we obtain for μ^0 and $\nu\neq\mu^0$

$$P(\xi'^{\mu^0}_j,\xi'^{\nu}_j) \quad = \quad \frac{1+a}{4}(\delta_{-\xi'^{\mu^0}_j+\xi'^{\nu}_j,2}+\delta_{-\xi'^{\mu}_j+\xi'^{\nu}_j,-2})+\frac{1-a}{4}\delta_{-\xi'^{\mu^0}_j+\xi'^{\nu}_j,0}. \tag{21}$$

Therefore, we obtain $\langle\xi'^{\mu_0}_j\xi'^{\nu}_j\rangle=-\langle\xi^{\mu_0}_j\xi^{\nu}_j\rangle=-a$ for $\mu_0\neq\nu$. Thus, the average over $\{\xi'\}$ is different from that over $\{\xi\}$ when $a\neq0$. Thus, we conclude that $(R_{1R},\cdots,-R_{\mu_0 R},\cdots,R_{pR},R_{1I},\cdots,-R_{\mu_0 I},\cdots,R_{pI})$ do not satisfy the SPEs for $a\neq0$. However, if all of the signs of $\{R_{\mu R}\}$ and $\{R_{\nu I}\}$ are reversed, these are also the solutions of the SPEs.

Now, we introduce the sublattice $\Lambda_l\,(l=1,\cdots,2^p)$, which is a set of i. In Λ_l, ξ^μ_i takes the value η^μ_l,

$$(\xi^1_i,\xi^2_i,\cdots,\xi^p_i)=(\eta^1_l,\eta^2_l,\cdots,\eta^p_l),\qquad i\in\Lambda_l.$$

$\{\eta^\mu_l\}$ are determined consecutively for $p\geq2$ as follows. When $p=2$, we define $\eta^1_1=1,\eta^2_1=1,\eta^1_2=1,\eta^2_2=-1$. Starting from this, other η^μ_l are determined. We set $\eta^1_l=1$ for $l=1,\cdots,2^{p-1}$. We define $\Lambda_{l+2^{p-1}}$ in which the following relations hold:

$$\eta^\mu_{l+2^{p-1}}=-\eta^\mu_l,\qquad(l=1,\cdots,2^{p-1}\ ,\ \ \mu=1,\cdots,p). \tag{22}$$

In addition, when the number of patterns is $p+1$, the values $\{\eta_l^{\mu,(p+1)}\}$ for $p+1$ are determined so that $\eta_l^{2,(p+1)}, \cdots, \eta_l^{p+1,(p+1)}$ have the following relationship with the values $\{\eta_l^{\mu,(p)}\}$ for p:

$$\eta_l^{\mu,(p+1)} = \eta_l^{\mu-1,(p)}, \qquad (l = 1, \cdots, 2^p \ , \quad \mu = 2, \cdots, p+1). \tag{23}$$

See Appendix C for details. For $j \in \Lambda_l$, Ξ_j takes the same value, which we denote by Ξ_l. Ξ_l is expressed as

$$\Xi_l = \sqrt{(\sum_{\mu=1}^{p} R_{\mu R}\eta_l^{\mu})^2 + (\sum_{\mu=1}^{p} R_{\mu I}\eta_l^{\mu})^2}, \tag{24}$$

$$\Xi_{l+2^{p-1}} = \Xi_l \ , (l = 1, 2, \cdots, 2^{p-1}). \tag{25}$$

Let P_l be the probability that ξ_i^{μ} is equal to η_l^{μ} for $i = 1, 2, \cdots, N$. By the self-averaging property, the average over N neurons is expressed as

$$\frac{1}{N}\sum_{j=1}^{N} g(\xi_j^{\mu}) = \sum_{l=1}^{2^p} P_l g(\eta_l^{\mu}). \tag{26}$$

The SPEs. (15) and (16) and Eq. (24) are rewritten as

$$R_{\mu R} = \beta J \sum_{\nu=1}^{p} c_{\mu\nu} R_{\nu R}, \tag{27}$$

$$R_{\mu I} = \beta J \sum_{\nu=1}^{p} c_{\mu\nu} R_{\nu I}, \tag{28}$$

$$c_{\mu\nu} \equiv \sum_{l=1}^{2^p} P_l u_l \eta_l^{\mu}\eta_l^{\nu} = c_{\nu\mu}, \tag{29}$$

$$u_l \equiv u(x_l) \, , \, x_l \equiv \beta J \Xi_l, \tag{30}$$

$$\Xi_l = \sqrt{R^2 + 2\sum_{\mu<\nu}\eta_l^{\mu}\eta_l^{\nu}(R_{\mu R}R_{\nu R} + R_{\mu I}R_{\nu I})}. \tag{31}$$

From Eq. (31), we obtain

$$\sum_{l=1}^{2^{p-1}} \Xi_l^2 = \sum_{l=1}^{2^{p-1}} \Big(R^2 + 2\sum_{\mu<\nu}\eta_l^{\mu}\eta_l^{\nu}(R_{\mu R}R_{\nu R} + R_{\mu I}R_{\nu I})\Big). \tag{32}$$

The following relation holds:

$$\sum_{l=1}^{2^{p-1}} \eta_l^{\mu}\eta_l^{\nu} = 2^{p-1}\delta_{\mu\nu}. \tag{33}$$

See Appendix B for its proof. Therefore, Eq. (32) is rewritten as follows:

$$\sum_{l=1}^{2^{p-1}} \Xi_l^2 = 2^{p-1} R^2,$$
$$R^2 = \frac{1}{2^{p-1}} \sum_{l=1}^{2^{p-1}} (\frac{x_l}{\beta J})^2. \quad (34)$$

From Eqs. (27) and (28), we obtain

$$R_\mu^2 = \beta J \sum_{\nu=1}^{p} c_{\mu\nu}(R_{\mu R} R_{\nu R} + R_{\mu I} R_{\nu I}). \quad (35)$$

Thus, by using Eq. (29), R^2 is expressed as

$$R^2 = \sum_{\mu=1}^{p} \beta J \sum_{\nu=1}^{p} c_{\mu\nu}(R_{\mu R} R_{\nu R} + R_{\mu I} R_{\nu I})$$
$$= \frac{2}{\beta J} \sum_{l=1}^{2^{p-1}} P_l u_l x_l^2. \quad (36)$$

2.2 Stable solutions of the SPEs and their stabilities

In this section, we list the stable solutions of the SPEs for $p \leq 3$. Detailed descriptions including unstable solutions are given in Appendix D. The stabilities of the stable solutions are analyzed in sect. 4 and those of the unstable solutions are analyzed in Appendix E.

2.2.1 Case of $p = 2$

	η_l^1	η_l^2
$l = 1$	1	1
$l = 2$	1	-1
$l = 3$	-1	-1
$l = 4$	-1	1

Table I: Values of $\{\eta_l^\mu\}$ in each sublattice for $p = 2$.

In Table I, we show the values of $\{\eta_l^\mu\}$ in each sublattice.

Memory pattern: M

$R_1 > 0$ and $R_2 = 0$. This solution exists only when there is no correlation between patterns. The solution is characterized as

$$u_1 = u_2 = \frac{1}{\beta J}, \qquad x_1 = x_2, \quad (37)$$

$$c_{\mu\nu} = \frac{1}{\beta J}\delta_{\mu\nu}, \quad (38)$$

$$R = R_1 = \frac{x_1}{\beta J}. \quad (39)$$

The critical temperature is $T_c^{(M)} = \frac{J}{2}$. The solution exists for $T < T_c^{(M)}$ and is stable.

Continuous attractor: CA

This solution exists for $a < 1$ and is characterized as

$$u_1 = \frac{1}{(1+a)\beta J}, \; u_2 = \frac{1}{(1-a)\beta J}, \quad (40)$$

$$c_{\mu\nu} = \frac{1}{\beta J}\delta_{\mu\nu}, \quad R^2 = \frac{x_1^2 + x_2^2}{2(\beta J)^2}. \quad (41)$$

The critical temperature is $T_c^{(CA)} = \frac{(1-a)J}{2}$. The CA is stable for $T < T_c^{(CA)}$.

Symmetric mixed solution: S$_1$ $(R_{1R} = R_{2R}, R_{1I} = R_{2I} = 0)$

This solution is characterized as

$$u_1 = \frac{1}{(1+a)\beta J}, \; u_2 = \frac{1}{2}, \; x_2 = 0, \quad (42)$$

$$c_{\mu\mu} = \frac{1}{2\beta J} + \frac{1-a}{4}, \; c_{\mu\nu} = \frac{1}{2\beta J} - \frac{1-a}{4} \quad (\mu \neq \nu), \quad (43)$$

$$R_1 = \frac{x_1}{2\beta J} = R_2, \; R = \frac{x_1}{\sqrt{2}\beta J}. \quad (44)$$

The solution exists for $T < T_c^{(S_1)} = \frac{(1+a)J}{2}$. The stability condition is

$$\frac{(1-a)J}{2} < T < T_c^{(S_1)}. \quad (45)$$

Thus, this solution is unstable for $a = 0$.

2.2.2 Case of $p = 3$

	η_l^1	η_l^2	η_l^3
$l = 1$	1	1	1
$l = 2$	1	1	-1
$l = 3$	1	-1	-1
$l = 4$	1	-1	1
$l = 5$	-1	-1	-1
$l = 6$	-1	-1	1
$l = 7$	-1	1	1
$l = 8$	-1	1	-1

Table II: Values of $\{\eta_l^\mu\}$ in each sublattice for $p = 3$.

In Table II, we show the values of $\{\eta_l^\mu\}$ in each sublattice for $p = 3$.

Memory pattern: M

This solution exists only when there is no correlation between patterns. It is characterized as

$$u_1 = u_2 = u_3 = u_4 = \frac{1}{\beta J}\,, \; x_1 = x_2 = x_3 = x_4, \tag{46}$$

$$c_{11} = \frac{1}{\beta J}\,, \; c_{12} = c_{13} = c_{23} = 0, \quad R = R_1 = \frac{x_1}{\beta J}. \tag{47}$$

This solution exists and is stable for $T < T_{\rm c}^{\rm (M)}$, where $T_{\rm c}^{\rm (M)} = \frac{J}{2}$.

Continuous attractor: CA

This solution exists for $a < \frac{1}{3}$ and is characterized as

$$u_1 = \frac{1}{(1+3a)\beta J}, \tag{48}$$

$$u_2 = u_3 = u_4 = \frac{1}{(1-a)\beta J}, \tag{49}$$

$$x_2 = x_3 = x_4, \tag{50}$$

$$c_{\mu\nu} = \frac{1}{\beta J}\delta_{\mu\nu}, \quad R_{2R} = R_{3R}, \; R^2 = \frac{x_1^2 + 3x_2^2}{4(\beta J)^2}. \tag{51}$$

We denote the critical point as $T_{\rm c}^{\rm (CA)}$, which is determined by the condition $x_1 = 3x_2$. For example, in the case of $a = 0.1$, $T_{\rm c}^{\rm (CA)}$=0.42. It is stable for $T < T_{\rm c}^{\rm (CA)}$.

Symmetric mixed solution: S$_4$ $(R_1 = R_2 = R_3)$

$R_{1R} = R_{2R} = R_{3R}$ holds, and this solution is characterized as

$$x_2 = x_3 = x_4 = \frac{x_1}{3}, \tag{52}$$

$$u_2 = u_3 = u_4, \tag{53}$$

$$\frac{1}{\beta J} = \frac{3}{4}(1+3a)u(x_1) + \frac{1}{4}(1-a)u(\frac{x_1}{3}), \tag{54}$$

$$c_{\mu\mu} = \frac{3}{\beta J} - 2(1+3a)u_1\,, \; c_{\mu\nu} = -\frac{1}{\beta J} + (1+3a)u_1 \quad (\mu \neq \nu), \tag{55}$$

$$R_1 = \frac{x_1}{3\beta J} = R_2 = R_3\,, \; R = \frac{x_1}{\sqrt{3}\beta J}. \tag{56}$$

The critical point is $T_{\rm c}^{\rm (S_4)} = \frac{(1+2a)J}{2}$. When $a < \frac{1}{3}$, this solution is stable for $T_{\rm c}^{\rm (CA)} < T < T_{\rm c}^{\rm (S_4)}$. When $a > \frac{1}{3}$, it is stable for $T < T_{\rm c}^{\rm (S_4)}$.

In Appendix D, we prove that for $a \neq 0$, when one or two of R_{1R}, R_{2R}, and R_{3R} have different signs, they do not satisfy the SPEs.

3. Characteristics and Conditions for the Existence of a Continuous Attractor

The CA is defined as a one-parameter family of solutions. The existence of the CA depends on p, J, β, and a.

3.1 Characteristics of the CA

The CA is characterized by $P_l u_l =$ constant for all l and $c_{\mu\nu} = \frac{1}{\beta J}\delta_{\mu\nu}$. Let us prove these statements.

(1) $P_l u_l =$ constant.

From Eqs. (34) and (36), we obtain

$$\sum_{l=1}^{2^{p-1}} x_l^2 \quad = \quad 2^p \beta J \sum_{l=1}^{2^{p-1}} P_l u_l x_l^2. \tag{57}$$

The sufficient condition for Eq. (57) is

$$x_l(1 - 2^p \beta J P_l u_l) = 0.$$

The condition satisfying this equation is either of the following two equations:

$$x_l \quad = \quad 0, \tag{58}$$

$$P_l u_l \quad = \quad \frac{1}{2^p \beta J}. \tag{59}$$

If Eq. (59) holds for all l, $P_l u_l$ is determined only by β, J, and p. Therefore, $x_1, \cdots, x_{2^{p-1}}$ is determined only by β, J, p, and a. In this case, if there is one variable that can change freely, it is the CA.

(2) $c_{\mu\nu} = \frac{1}{\beta J}\delta_{\mu\nu}$

Now, let us assume that $P_l u_l =$ constant for all l. Then, by using $\sum_{l=1}^{2^{p-1}} \eta_l^\mu \eta_l^\nu = 2^{p-1}\delta_{\mu\nu}$, we obtain

$$c_{\mu\nu} \quad = \quad P_l u_l \sum_{l=1}^{2^p} \eta_l^\mu \eta_l^\nu = P_l u_l 2^p \delta_{\mu\nu}.$$

Therefore, because $P_l u_l = \frac{1}{2^p \beta J}$, we derive

$$c_{\mu\nu} = \frac{1}{\beta J}\delta_{\mu\nu}. \tag{60}$$

Conversely, if Eq. (60) holds, the SPEs. (27) and (28) are satisfied and $x_1, \cdots, x_p$ are determined by Eq. (59).

3.2 Conditions for the existence of the CA for $a = 0$

The CA exists for arbitrary p (≥ 2). Let us prove this. Let us assume that only two R_μ are not zero. For example, we assume $R_{1R} \neq 0$, $R_{1I} = 0$, $R_2 \neq 0$, $R_3 = \cdots = R_p = 0$. This is possible since there is no correlation between patterns. From Eq. (59), since $P_l = 1/2^p$, $u_1 = u(x_1) = \frac{1}{\beta J}$. Thus, the solution exists for $u(0) \geq \frac{1}{\beta J}$. This implies that $T_\mathrm{c}^{(\mathrm{CA})} = Ju(0) = \frac{J}{2}$.

3.3 Conditions for the existence of the CA for $a > 0$

The condition on p for the existence of the CA is obtained by comparing the number of conditions for the CA and the number of variables $R_{\mu R}$ and $R_{\mu I}$. The number of conditions is the number of equations on Ξ_l, and is 2^{p-1} since $\Xi_{l+2^{p-1}} = \Xi_l$ holds. Because of the rotational symmetry, $R_{1I} = 0$ can be assumed. The CA is assumed to be a one-parameter family. Therefore, the number of dependent variables that should be decided is $2(p-1)$. Thus, $2^{p-1} = 2(p-1)$ is the condition on p for the existence of the CA. Only $p = 2$ and 3 satisfy this condition. Thus, the CA does not exist for $p > 3$. The critical point $T_\mathrm{c}^{(\mathrm{CA})}$ of the solution for $p = 2$ is obtained from Eq. (40) for $u(x_2)$,

$$\frac{1}{(1-a)\beta J} \leq \frac{1}{2}.$$

Therefore, the critical point is $T_\mathrm{c}^{(\mathrm{CA})} = \frac{(1-a)J}{2}$. In the case of $p = 3$, $x_1 < 3x_2$ is necessary. When $x_1 = 3x_2$, the CA coincides with the symmetric mixed solution S_4. See Appendices D and E for details. When the CA disappears, the symmetric mixed solution S_4 becomes stable.

Now, for $p = 3$, we derive the condition on the correlation a for the existence of the CA. When $T \sim 0$, the function u_l becomes very small from Eqs. (48) and (49), and x_l becomes very large. The function $u(x)$ can be approximated for $x \gg 1$ as follows:

$$u(x) \simeq \frac{1}{x}.$$

See Appendix B. Since $u_1 = \frac{1}{(1+3a)\beta J}$ and $u_2 = \frac{1}{(1-a)\beta J}$, we obtain

$$\begin{aligned} x_1 &\simeq \frac{(1+3a)J}{T}, \\ x_2 &\simeq \frac{(1-a)J}{T}. \end{aligned}$$

Substituting them into the condition for the existence of the CA, i.e., $x_1 < 3x_2$, we obtain

$$a \quad < \quad \frac{1}{3}.$$

In Fig. 3, we show the phase diagram in the (a, T) plane for $p = 2$ and 3. The theoretical results agree with the numerical results obtained by MCMCs reasonably well.

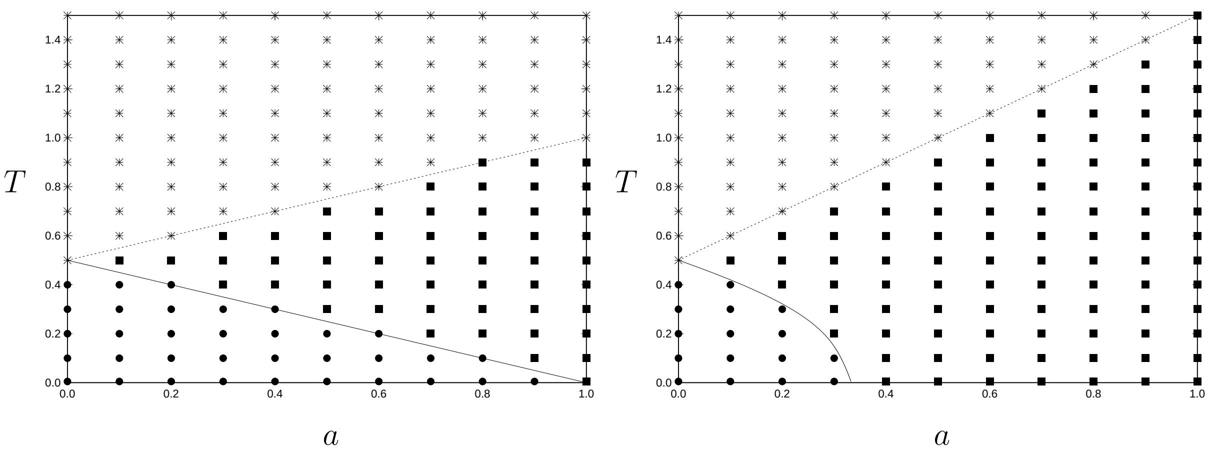

Fig. 3. Phase diagram of the CA and the symmetric mixed solutions S_1 for $p = 2$ and S_4 for $p = 3$ in the (a, T) plane. Curves: theoretical results. Solid curve: $T_c^{(CA)}$, dotted curves: $T_c^{(S_1)}$ and $T_c^{(S_4)}$. Symbols: results from MCMCs with $N = 20000$. Circles: CA, squares: S_1, S_4, stars: Para. Left: $p = 2$, right: $p = 3$.

4. Stabilities of Relevant Solutions for $p \leq 3$

In this section, we study the stabilities of relevant solutions of the SPEs. Those for unstable solutions are given in Appendix E. We calculate the Hessian of the free energy F. The components of the Hessian matrix $\mathcal{H}$ are written as follows:

$$\mathcal{H}_{(\mu R, \nu R)} \equiv \frac{\partial^2 F}{\partial R_{\mu R} \partial R_{\nu R}} = JN\Big(\delta_{\mu\nu} - \beta J c_{\mu\nu} - (\beta J)^3 \sum_{l=1}^{2^p} P_l u_l X_l \eta_l^{\mu} \eta_l^{\nu} (\zeta_{lR})^2\Big), \quad (61)$$

$$\mathcal{H}_{(\mu I, \nu I)} \equiv \frac{\partial^2 F}{\partial R_{\mu I} \partial R_{\nu I}} = JN\Big(\delta_{\mu\nu} - \beta J c_{\mu\nu} - (\beta J)^3 \sum_{l=1}^{2^p} P_l u_l X_l \eta_l^{\mu} \eta_l^{\nu} (\zeta_{lI})^2\Big), \quad (62)$$

$$\mathcal{H}_{(\mu R, \nu I)} \equiv \frac{\partial^2 F}{\partial R_{\mu R} \partial R_{\nu I}} = JN\Big(-(\beta J)^3 \sum_{l=1}^{2^p} P_l u_l X_l \eta_l^{\mu} \eta_l^{\nu} \zeta_{lR} \zeta_{lI}\Big), \quad (63)$$

where

$$\zeta_{lR} \equiv \sum_{\omega=1}^{p} R_{\omega R} \eta_l^{\omega}, \quad \zeta_{lI} \equiv \sum_{\omega=1}^{p} R_{\omega I} \eta_l^{\omega},$$

$$x_l = \beta J \Xi_l = \beta J\sqrt{(\zeta_{lR})^2 + (\zeta_{lI})^2}, \quad X_l \equiv \frac{u'(x_l)}{x_l u(x_l)}.$$

These are general expressions for the Hessian matrix.

4.1 Case of $p = 2$

Memory pattern

The memory pattern exists only when $a = 0$. Since $R_{1I} = 0$, we obtain

$$R_{2R} = R_{2I} = 0.$$

The values of x_l, u_l, and R for the memory pattern are

$$x_1 = x_2, \quad u_1 = u_2 = \frac{1}{\beta J}, \quad R = \frac{x_1}{\beta J}.$$

The solution exists for $u_1 \leq \frac{1}{2}$. Thus, the critical point is $T_c^{(M)} = \frac{J}{2}$. The values of ζ_{lR}, ζ_{lI}, $c_{\mu\mu}, c_{\mu\nu}$, and $P_l u_l$ are given as

$$\zeta_{1R} = \zeta_{2R} = R_{1R}, \quad \zeta_{1I} = \zeta_{2I} = 0,$$
$$c_{\mu\mu} = \frac{1}{\beta J}, \quad c_{\mu\nu} = 0 \quad (\mu \neq \nu), \quad P_l u_l = \frac{1}{2^p \beta J}.$$

Therefore, the components of the Hessian matrix $\mathcal{H}$ are

$$\begin{aligned} \mathcal{H}_{1R1R} &= -\frac{1}{2} JN(\beta J)^2 (\zeta_{1R})^2 (X_1 + X_2) \\ &= -JN(\beta J)^2 (\zeta_{1R})^2 X_1 = \mathcal{H}_{2R2R} \equiv A, \\ \mathcal{H}_{\mu R\nu R} &= 0, \quad (\mu \neq \nu), \quad \mathcal{H}_{\mu R\nu I} = \mathcal{H}_{\mu I\nu I} = 0 \quad (\mu, \nu = 1, 2). \end{aligned}$$

We define the arrangement of the matrix elements as $1R, 2R, 1I$, and $2I$.

$$\mathcal{H} = \begin{array}{c} \\ 1R \\ 2R \\ 1I \\ 2I \end{array} \begin{array}{c} \begin{array}{cccc} 1R & 2R & 1I & 2I \end{array} \\ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

The four eigenvalues of this matrix are

$$\lambda = 0 \ \ \text{(2-fold)}, \ A \ \ \text{(2-fold)}.$$

A is expressed as

$$A = -JN(\beta J)^2 (\zeta_{1R})^2 X_1.$$

Since $J > 0, N > 0$ and $X_l < 0$, this is positive. Thus, the Hessian matrix $\mathcal{H}$ at the memory pattern has zero (2-fold) and positive (2-fold) eigenvalues. Thus, it is stable.

The continuous attractor

By using the relations $P_l u_l = \frac{1}{2^p \beta J}$ and $c_{\mu\nu} = \frac{1}{\beta J}\delta_{\mu\nu}$ for the CA, the components of the Hessian matrix are given by

$$\frac{\partial^2 F}{\partial R_{\mu R}\partial R_{\nu R}} = -JN(\beta J)^2 \frac{1}{2^{p-1}} \sum_{l=1}^{2^{p-1}} X_l \eta_l^\mu \eta_l^\nu (\zeta_{lR})^2, \tag{64}$$

$$\frac{\partial^2 F}{\partial R_{\mu I}\partial R_{\nu I}} = -JN(\beta J)^2 \frac{1}{2^{p-1}} \sum_{l=1}^{2^{p-1}} X_l \eta_l^\mu \eta_l^\nu (\zeta_{lI})^2, \tag{65}$$

$$\frac{\partial^2 F}{\partial R_{\mu R}\partial R_{\nu I}} = -JN(\beta J)^2 \frac{1}{2^{p-1}} \sum_{l=1}^{2^{p-1}} X_l \eta_l^\mu \eta_l^\nu \zeta_{lR}\zeta_{lI}. \tag{66}$$

<u>Case of $a = 0$</u>

We investigate the stability of the CA for $a = 0$. For $l = 1, \cdots, p$, we have the following relations:

$$P_l u_l = \frac{1}{2^p \beta J}, \quad u_l = \frac{1}{\beta J} > 0, \quad x_l = \text{constant} > 0, \quad X_l = \frac{u'(x_l)}{x_l u(x_l)} < 0.$$

None of the quantities depend on l. We define Λ as $\Lambda = -\frac{1}{JN(\beta J)^2}\mathcal{H}$. Therefore, we obtain

$$\Lambda_{\mu R \nu R} \equiv -\frac{1}{JN(\beta J)^2}\frac{\partial^2 F}{\partial R_{\mu R}\partial R_{\nu R}} = \frac{1}{2^{p-1}} X \sum_{l=1}^{2^{p-1}} \eta_l^\mu \eta_l^\nu (\zeta_{lR})^2, \tag{67}$$

$$\Lambda_{\mu R \nu I} = \frac{1}{2^{p-1}} X \sum_{l=1}^{2^{p-1}} \eta_l^\mu \eta_l^\nu \zeta_{lR}\zeta_{lI}, \tag{68}$$

$$\Lambda_{\mu I \nu I} = \frac{1}{2^{p-1}} X \sum_{l=1}^{2^{p-1}} \eta_l^\mu \eta_l^\nu (\zeta_{lI})^2, \tag{69}$$

where $X \equiv X_l$. For $p \geq 2$, we assume $R_{1R} \neq 0$, $R_{1I} = 0$, $R_2 \neq 0$, $R_3 = \cdots = R_p = 0$ without loss of generality. As is shown in Appendix D, for $a = 0$ and $p = 2$, when we assume $R_{2I} \neq 0, R_{2R} = 0$ follows. Then, we have

$$R_1 = |R_{1R}| \quad , \quad R_2 = |R_{2I}|,$$

$$\zeta_{lR} = \sum_{\mu=1}^{p} R_{\mu R}\eta_l^\mu = R_{1R}\eta_l^1 \quad , \quad \zeta_{lI} = \sum_{\mu=1}^{p} R_{\mu I}\eta_l^\mu = R_{2I}\eta_l^2.$$

We substitute these into Eqs. (67)-(69). The following equation is verified:

$$\sum_{l=1}^{2^{p-1}} \eta_l^\mu \eta_l^\nu \eta_l^1 \eta_l^2 = \begin{cases} 2^{p-1} & (\mu,\nu) = (1,2) \ \text{ or } \ (2,1), \\ 0 & \text{other cases.} \end{cases} \tag{70}$$

See Appendix B for the proof. First of all, we consider the case when $(\mu,\nu)=(1,2)$ or $(2,1)$. Because $\sum_{l=1}^{2^{p-1}}(\eta_l^1\eta_l^2)^2=2^{p-1}$,

$$\Lambda_{\mu R\nu I} = \frac{1}{2^{p-1}}XR_{1R}R_{2I}\sum_{l=1}^{2^{p-1}}\eta_l^\mu\eta_l^\nu\eta_l^1\eta_l^2 = XR_{1R}R_{2I}.$$

When $(\mu,\nu)\neq(1,2),(2,1)$,

$$\Lambda_{\mu R\nu I} = \frac{1}{2^{p-1}}XR_{1R}R_{2I}\sum_{l=1}^{2^{p-1}}\eta_l^\mu\eta_l^\nu\eta_l^1\eta_l^2 = 0.$$

Thus, each component of the matrix Λ is expressed as follows:

$$\begin{aligned}
\Lambda_{\mu R\nu R} &= \frac{1}{2^{p-1}}XR_1^2\sum_{l=1}^{2^{p-1}}\eta_l^\mu\eta_l^\nu = XR_1^2\delta_{\mu\nu},\\
\Lambda_{\mu R\nu I} &= \frac{1}{2^{p-1}}XR_{1R}R_{2I}\sum_{l=1}^{2^{p-1}}\eta_l^\mu\eta_l^\nu\eta_l^1\eta_l^2\\
&= \begin{cases} XR_{1R}R_{2I} & (\mu,\nu)=(1,2) \text{ or } (2,1),\\ 0 & \text{other cases,}\end{cases}\\
\Lambda_{\mu I\nu I} &= XR_2^2\delta_{\mu\nu}.
\end{aligned}$$

The matrix Λ is

$$\Lambda = \begin{matrix} & 1R & 1I & 2R & 2I \\ \begin{matrix}1R\\1I\\2R\\2I\end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} \Lambda_{1R1R} & \Lambda_{1R1I} & \Lambda_{1R2R} & \Lambda_{1R2I}\\ \Lambda_{1I1R} & \Lambda_{1I1I} & \Lambda_{1I2R} & \Lambda_{1I2I}\\ \Lambda_{2R1R} & \Lambda_{2R1I} & \Lambda_{2R2R} & \Lambda_{2R2I}\\ \Lambda_{2I1R} & \Lambda_{2I1I} & \Lambda_{2I2R} & \Lambda_{2I2I}\end{pmatrix}}\end{matrix} = X\begin{pmatrix} R_1^2 & 0 & 0 & R_{1R}R_{2I}\\ 0 & R_2^2 & R_{1R}R_{2I} & 0\\ 0 & R_{1R}R_{2I} & R_1^2 & 0\\ R_{1R}R_{2I} & 0 & 0 & R_2^2\end{pmatrix}.$$

We solve the eigenvalue problem of this matrix as

$$\begin{aligned}
|\Lambda-\lambda E| &= \begin{vmatrix} XR_1^2-\lambda & 0 & 0 & XR_{1R}R_{2I}\\ 0 & XR_2^2-\lambda & XR_{1R}R_{2I} & 0\\ 0 & XR_{1R}R_{2I} & XR_1^2-\lambda & 0\\ XR_{1R}R_{2I} & 0 & 0 & XR_2^2-\lambda\end{vmatrix}\\
&= \lambda^2(\lambda-XR^2)^2=0.
\end{aligned}$$

The eigenvalues are obtained as

$$\lambda_1=0 \ \ \text{(2-fold)},\qquad \lambda_2=XR^2<0 \ \ \text{(2-fold)}.$$

Thus, the eigenvalues of the Hessian matrix $\mathcal{H}$ are zero and $-JN(\beta J)^2 XR^2 > 0$. Therefore, the CA is stable. The free energy of the CA has the shape of a valley, which is composed of the route from a certain memory pattern to another memory pattern. The eigenvalue with twofold degeneracy $\lambda_1 = 0$ reflects the existence of the CA and the rotational symmetry.

Case of $a > 0$

If there is a correlation between patterns, all overlaps R_μ have nonzero values. Therefore, we assume $R_{1R} > 0, R_{1I} = 0$, and $R_2 \neq 0$ without loss of generality. Since $R_{1I} = 0$, we obtain

$$\zeta_{1R} = R_{1R} + R_{2R}, \quad \zeta_{2R} = R_{1R} - R_{2R},$$

$$\zeta_{1I} = R_{1I} + R_{2I} = R_{2I}, \quad \zeta_{2I} = R_{1I} - R_{2I} = -R_{2I}.$$

The Hessian matrix $\mathcal{H}$ is obtained from Eqs. (64)-(66). Λ is defined as

$$\Lambda = -\frac{2}{JN(\beta J)^2}\mathcal{H}.$$

We obtain

$$\begin{aligned}
\Lambda_{1R1R} &= X_1(\zeta_{1R})^2 + X_2(\zeta_{2R})^2 = \Lambda_{2R2R} \equiv A < 0, \\
\Lambda_{1R2R} &= X_1(\zeta_{1R})^2 - X_2(\zeta_{2R})^2 = \Lambda_{2R1R} \equiv B, \\
\Lambda_{1I1I} &= X_1(\zeta_{1I})^2 + X_2(\zeta_{2I})^2 = \Lambda_{2I2I} \equiv C < 0, \\
\Lambda_{1I2I} &= X_1(\zeta_{1I})^2 - X_2(\zeta_{2I})^2 = \Lambda_{2I1I} \equiv D, \\
\Lambda_{1R1I} &= X_1\zeta_{1R}\zeta_{1I} + X_2\zeta_{2R}\zeta_{2I} = \Lambda_{2R2I} \equiv G, \\
\Lambda_{1R2I} &= X_1\zeta_{1R}\zeta_{1I} - X_2\zeta_{2R}\zeta_{2I} = \Lambda_{2R1I} \equiv K.
\end{aligned}$$

The matrix Λ is

$$\Lambda = \begin{array}{c} \\ 1R \\ 1I \\ 2R \\ 2I \end{array} \begin{array}{c} \begin{array}{cccc} 1R & 1I & 2R & 2I \end{array} \\ \begin{pmatrix} A & G & B & K \\ G & C & K & D \\ B & K & A & G \\ K & D & G & C \end{pmatrix} \end{array}.$$

From the rotational symmetry, we can omit the row and column that contain R_{1I}. We call this matrix Λ again and solve the eigenvalue problem of Λ,

$$|\Lambda - \lambda E| = \begin{vmatrix} A-\lambda & B & K \\ B & A-\lambda & G \\ K & G & C-\lambda \end{vmatrix} = 0,$$

$$\begin{aligned} \lambda^3 - (2A+C)\lambda^2 &+ (2AC + A^2 - B^2 - G^2 - K^2)\lambda \\ &- (A^2C + 2BGK - AK^2 - B^2C - A^2G) = 0. \end{aligned}$$

The constant term becomes 0 and thus there is an eigenvalue of 0. Thus, we obtain

$$\lambda^2 - (2A+C)\lambda + 2AC + A^2 - B^2 - G^2 - K^2 = 0.$$

By defining $g \equiv -(2A+C)$ and $h \equiv 2AC+A^2-B^2-G^2-K^2$, we obtain $\lambda^2+g\lambda+h = 0$. The solutions are

$$\lambda_\pm = \frac{1}{2}(-g \pm \sqrt{g^2-4h}).$$

g^2 and h are calculated as

$$\begin{aligned} g^2 &= \Big(X_1\{2(\zeta_{1R})^2 + (\zeta_{1I})^2\} + X_2\{2(\zeta_{2R})^2 + (\zeta_{2I})^2\}\Big)^2, \\ h &= 2X_1X_2\Big((\zeta_{1R})^2(\zeta_{2I})^2 + (\zeta_{1I})^2(\zeta_{2R})^2 + 2(\zeta_{1R})^2(\zeta_{2R})^2\Big). \end{aligned}$$

Since $A < 0$ and $C < 0$, $g > 0$ follows. In addition, since $X_l < 0$, $h > 0$ follows. Next we show that $g^2 - 4h$ is positive.

$$\begin{aligned} g^2 - 4h &= X_1^2\{2(\zeta_{1R})^2 + (\zeta_{1I})^2\}^2 + X_2^2\{2(\zeta_{2R})^2 + (\zeta_{2I})^2\}^2 \\ &\quad + 2X_1X_2\{(\zeta_{1I})^2(\zeta_{2I})^2 - 2(\zeta_{1R})^2(\zeta_{2I})^2 - 2(\zeta_{1I})^2(\zeta_{2R})^2 - 4(\zeta_{1R})^2(\zeta_{2R})^2\}. \end{aligned}$$

By defining z_1, z_2, and z_3 as

$$\begin{aligned} z_1 &= \{2(\zeta_{1R})^2 + (\zeta_{1I})^2\}^2, \\ z_2 &= (\zeta_{1I})^2(\zeta_{2I})^2 - 2(\zeta_{1R})^2(\zeta_{2I})^2 - 2(\zeta_{1I})^2(\zeta_{2R})^2 - 4(\zeta_{1R})^2(\zeta_{2R})^2, \\ z_3 &= \{2(\zeta_{2R})^2 + (\zeta_{2I})^2\}^2, \end{aligned}$$

g^2-4h is expressed as $g^2-4h = z_1X_1^2+2z_2X_2X_1+z_3X_2^2$. Since $z_1 > 0$, if the discriminant d of this quadratic formula for X_1 is negative, $g^2 - 4h > 0$ follows.

$$d = (z_2X_2)^2 - z_1z_3X_2^2 = X_2^2(z_2^2 - z_1z_3).$$

We put $\tilde{z}_1 = 2(\zeta_{1R})^2 + (\zeta_{1I})^2$ and $\tilde{z}_3 = 2(\zeta_{2R})^2 + (\zeta_{2I})^2$, and obtain

$$z_2^2 - z_1 z_3 = (z_2 + \tilde{z}_1\tilde{z}_3)(z_2 - \tilde{z}_1\tilde{z}_3).$$

Each factor is calculated as

$$\begin{aligned} z_2 + \tilde{z}_1\tilde{z}_3 &= 2(\zeta_{1I})^2(\zeta_{2I})^2 > 0, \\ z_2 - \tilde{z}_1\tilde{z}_3 &= -4(\zeta_{1R})^2(\zeta_{2I})^2 - 4(\zeta_{1I})^2(\zeta_{2R})^2 - 8(\zeta_{1R})^2(\zeta_{2R})^2 < 0. \end{aligned}$$

Thus, the discriminant is negative and we obtain $g^2 - 4h > 0$. Therefore, two eigenvalues $\lambda_\pm$ of Λ are negative. Thus, the Hessian matrix $\mathcal{H}$ at the CA has zero (2-fold) and two positive eigenvalues. This implies that the free energy of the CA has the shape of a valley and the CA is stable.

Symmetric mixed solution: $\mathbf{S}_1$

We assume $R_{1I} = 0$ from the rotational symmetry. Thus, we obtain

$$R_{1R} = R_{2R}\ ,\ \ R_{2I} = 0.$$

The values of u_l , R_{lR}, R_{lI}, and R are

$$u_1 = \frac{1}{(1+a)\beta J}\ ,\quad u_2 = \frac{1}{2}, R_{1R} = \frac{x_1}{2\beta J} = R_{2R},\quad R = \frac{x_1}{\sqrt{2}\beta J}.$$

Thus, the critical point is $T_c^{(S_1)} = \frac{(1+a)J}{2}$. The values of $c_{\mu\mu}$ and $c_{\mu\nu}$ are

$$c_{\mu\mu} = \frac{1}{2\beta J} + \frac{1-a}{4},\quad c_{\mu\nu} = \frac{1}{2\beta J} - \frac{1-a}{4},\qquad (\mu \neq \nu).$$

Thus, we obtain

$$\delta_{\mu\nu} - \beta J c_{\mu\nu} = \begin{cases} \frac{1}{2} - \frac{1-a}{4}\beta J, & (\mu = \nu), \\ -\frac{1}{2} + \frac{1-a}{4}\beta J, & (\mu \neq \nu). \end{cases}$$

Putting $\gamma \equiv JN(\frac{1}{2} - \frac{1-a}{4}\beta J)$, the Hessian matrix $\mathcal{H}$ is expressed as

$$\mathcal{H} = \begin{array}{c} \\ 1R \\ 2R \\ 1I \\ 2I \end{array} \begin{array}{c} \begin{array}{cccc} 1R & 2R & 1I & 2I \end{array} \\ \begin{pmatrix} A & A-2\gamma & 0 & 0 \\ A-2\gamma & A & 0 & 0 \\ 0 & 0 & \gamma & -\gamma \\ 0 & 0 & -\gamma & \gamma \end{pmatrix} \end{array},$$

where $A = \gamma - 2JN(\beta J)^2 X_1 R_{1R}^2$. Its determinant is

$$|\mathcal{H} - \lambda E| = (2A - 2\gamma - \lambda)(2\gamma - \lambda)^2(-\lambda).$$

The eigenvalues of this matrix are the following:

$$\lambda = 0,\ 2(A-\gamma),\ 2\gamma\ \text{(2-fold)}.$$

Let us study the signs of the eigenvalues. We have

$$2(A-\gamma) \quad = \quad -2JN(\beta J)^2 X_1 R_{1R}^2.$$

Since $X_l < 0$, this is positive. Thus, if γ is positive, the solution is stable. The condition for this is

$$T \quad > \quad \frac{(1-a)J}{2}.$$

Therefore, the symmetric mixed solution S_1 is stable for $T > \frac{(1-a)J}{2}$.

4.2 $p \geq 3$

Memory pattern: M

Firstly, we study the case of $p = 3$. The memory pattern exists only when $a = 0$. We assume $R_{1I} = 0$ from the rotational symmetry. Thus, we obtain

$$R_{2R} = R_{2I} = R_{3R} = R_{3I} = 0.$$

The values of u_l and R are

$$x_1 = x_2 = x_3 = x_4, \tag{71}$$

$$u_1 = u_2 = \frac{1}{\beta J}, \tag{72}$$

$$R = \frac{x_1}{\beta J}. \tag{73}$$

From Eq. (72), the critical point is $T_{\mathrm{c}}^{(\mathrm{M})} = \frac{J}{2}$. The values of $c_{\mu\mu}, c_{\mu\nu}$, and $P_l u_l$ are

$$c_{\mu\mu} = \frac{1}{\beta J},\quad c_{\mu\nu} = 0, \quad (\mu \neq \nu),\quad P_l u_l = \frac{1}{2^p \beta J}.$$

Then, we have

$$\delta_{\mu\nu} - \beta J c_{\mu\nu} = 0,\ \text{for any}\ \mu, \nu.$$

In this solution, $X_l = \frac{u'(x_l)}{x_l u(x_l)} = X_1$. Therefore, the Hessian matrix $\mathcal{H}$ is expressed as

$$\mathcal{H} = \begin{array}{c} \\ 1R \\ 2R \\ 3R \\ 1I \\ 2I \\ 3I \end{array} \begin{array}{c} \begin{array}{cccccc} 1R & 2R & 3R & 1I & 2I & 3I \end{array} \\ \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

where $A = -JN(\beta J)^2 R_1^2 X_1$. The eigenvalues of this matrix are

$$\lambda = A \text{ (3-fold)}, \ 0 \text{ (3-fold)}.$$

Since $J > 0, N > 0$ and $X_l < 0$, we obtain

$$A = -JN(\beta J)^2 R_1^2 X_1 > 0.$$

Thus, the Hessian matrix $\mathcal{H}$ at the memory pattern has zero (3-fold) and three degenerate positive eigenvalues. Thus, the memory pattern is stable.

Now, let us consider the case of $p > 3$. In this case, since $R_{1R} \neq 0$ and the other $R_{\mu R}$ and $R_{\nu I}$ are zero, we have

$$\mathcal{H}_{\mu R \nu R} = A\delta_{\mu\nu}, \tag{74}$$

$$\mathcal{H}_{\mu R \nu I} = \mathcal{H}_{\mu I \nu I} = 0, \quad (\mu, \nu = 1, \cdots, p). \tag{75}$$

Thus, $\mathcal{H}$ has p-fold zero eigenvalues and p degenerate positive eigenvalues, A. This is because the memory pattern is the end point of $p-1$ different CAs and thus it has $p-1$ zero eigenvalues and another zero eigenvalue due to the rotational symmetry. Therefore, the memory pattern is stable for any p when $a = 0$.

Continuous attractor: CA

Case of $a = 0$

Similarly to the case of $p = 2$, the matrix $\Lambda = -\frac{\mathcal{H}}{JN(\beta J)^2}$ for $p > 2$ is given as

$$\mathcal{H} = X \begin{pmatrix} R_1^2 & 0 & 0 & R_{1R}R_{2I} & 0 & 0 & \cdots & 0 & 0 \\ 0 & R_2^2 & R_{1R}R_{2I} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & R_{1R}R_{2I} & R_1^2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ R_{1R}R_{2I} & 0 & 0 & R_2^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & R_1^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_2^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & R_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_2^2 \end{pmatrix}.$$

We solve the eigenvalue problem of this matrix as

$$|\Lambda - \lambda E| = \lambda^2(\lambda - XR^2)^2(XR_2^2 - \lambda)^{p-2}(XR_1^2 - \lambda)^{p-2} = 0.$$

The eigenvalues of the Hessian matrix are zero (2-fold), $-JN(\beta J)^2 XR^2 > 0$ (2-fold), $-JN(\beta J)^2 XR_1^2 > 0$ (($p-2$)-fold) and $-JN(\beta J)^2 XR_2^2 > 0$ (($p-2$)-fold). Therefore, the free energy of the CA has the shape of a valley and the CA is stable.

Case of $a > 0$

Since the CA does not exist for $p > 3$, we consider the case of $p = 3$. It is proved that $R_{1R} = R_{2R} = R_{3R} > 0$ can be assumed (see Appendix F). Now, we define a', b', and c' as

$$\begin{aligned} a' &= R_{1R}R_{2R} + R_{1I}R_{2I}, \\ b' &= R_{1R}R_{3R} + R_{1I}R_{3I}, \\ c' &= R_{2R}R_{3R} + R_{2I}R_{3I}. \end{aligned}$$

In Appendix D, we prove that

$$a' = \frac{\Xi_1^2 - \Xi_2^2}{8}.$$

Then, from $R_{1R} = R_{2R} = R_{3R}$ and $R_{1R}^2 = a'$, we obtain

$$\begin{aligned} R_{1R}^2 &= \frac{\Xi_1^2 - \Xi_2^2}{8} = R_{2R} = R_{3R}, \\ R_{2I}^2 &= \frac{1}{2}\{-(R_{1R}^2 + 2R_{2R}^2 - R^2) + \sqrt{(R_{1R}^2 + 2R_{2R}^2 - R^2)^2 - 4(a' - R_{2R}^2)^2}\} \\ &= -(R_{1R}^2 + 2R_{2R}^2 - R^2) = R^2 - 3a' = \frac{9\Xi_2^2 - \Xi_1^2}{8}, \\ R_{3I}^2 &= \frac{1}{2}\{-(R_{1R}^2 + 2R_{2R}^2 - R^2) - \sqrt{(R_{1R}^2 + 2R_{2R}^2 - R^2)^2 - 4(a' - R_{2R}^2)^2}\} = 0. \end{aligned}$$

In Appendix F, $\Xi_1 \leq 3\Xi_2$ is derived in order to show that $R_{2I}^2 \geq 0$ holds. Furthermore, since $R_{1I} = 0$, the values of ζ_{lR} and ζ_{lI} are

$$\begin{aligned} &\zeta_{1R} = 3R_{1R}, \quad \zeta_{2R} = \zeta_{4R} = R_{1R}, \quad \zeta_{3R} = -R_{1R}, \\ &\zeta_{1I} = \zeta_{2I} = R_{2I}, \quad \zeta_{3I} = \zeta_{4I} = -R_{2I}. \end{aligned}$$

For the CA, $X_2 = X_3 = X_4$ follows from $u_2 = u_3 = u_4$. The Hessian matrix $\mathcal{H}$ is obtained from Eqs. (64)-(66). We define Λ as

$$\Lambda = -\frac{4}{JN(\beta J)^2}\mathcal{H}.$$

The components of Λ are

$$\begin{aligned} \Lambda_{1R1R} &= 3(3X_1 + X_2)R_{1R}^2 = \Lambda_{2R2R} = \Lambda_{3R3R} \equiv A, \\ \Lambda_{1R2R} &= (9X_1 - X_2)R_{1R}^2 = \Lambda_{2R1R} \\ &= \Lambda_{1R3R} = \Lambda_{3R1R} = \Lambda_{2R3R} = \Lambda_{3R2R} \equiv B, \\ \Lambda_{1I1I} &= (X_1 + 3X_2)R_{2I}^2 = \Lambda_{2I2I} = \Lambda_{3I3I} \equiv C, \end{aligned}$$

$$
\begin{aligned}
\Lambda_{1I2I} &= (X_1 - X_2)R_{2I}^2 = \Lambda_{2I1I} \\
&= \Lambda_{1I3I} = \Lambda_{3I1I} = \Lambda_{2I3I} = \Lambda_{3I2I} \equiv D, \\
\Lambda_{1R1I} &= (3X_1 + X_2)R_{1R}R_{2I} = \Lambda_{2R2I} = \Lambda_{3R3I} \\
&= \Lambda_{1R2I} = \Lambda_{2R1I} = \Lambda_{2R3I} = \Lambda_{3R2I} \equiv E, \\
\Lambda_{1R3I} &= 3(X_1 - X_2)R_{1R}R_{2I} = \Lambda_{3R1I} \equiv G.
\end{aligned} \tag{76}
$$

We rewrite these components as

$$
\begin{aligned}
A &= 3(3X_1 + X_2)R_{1R}^2, \\
B &= (9X_1 - X_2)R_{1R}^2 = \frac{9X_1 - X_2}{3(3X_1 + X_2)}A = \gamma A, \\
C &= (X_1 + 3X_2)R_{2I}^2, \\
D &= (X_1 - X_2)R_{2I}^2 = \frac{X_1 - X_2}{X_1 + 3X_2}C = \omega C, \\
E &= (3X_1 + X_2)R_{1R}R_{2I}, \\
G &= 3(X_1 - X_2)R_{1R}R_{2I} = \frac{3(X_1 - X_2)}{3X_1 + X_2}E = \epsilon E,
\end{aligned}
$$

where

$$
\gamma = \frac{9X_1 - X_2}{3(3X_1 + X_2)}, \quad \omega = \frac{X_1 - X_2}{X_1 + 3X_2}, \quad \epsilon = \frac{3(X_1 - X_2)}{3X_1 + X_2}.
$$

Owing to the rotational symmetry, the row and column that contain R_{1I} can be omitted. We call this matrix Λ again,

$$
\Lambda = \begin{array}{c} \\ 1R \\ 2R \\ 3R \\ 2I \\ 3I \end{array}
\begin{array}{c} \begin{array}{ccccc} 1R & 2R & 3R & 2I & 3I \end{array} \\
\begin{pmatrix}
A & \gamma A & \gamma A & E & \epsilon E \\
\gamma A & A & \gamma A & E & E \\
\gamma A & \gamma A & A & E & E \\
E & E & E & C & \omega C \\
\epsilon E & E & E & \omega C & C
\end{pmatrix} \end{array}.
$$

We solve the eigenvalue problem of the reduced matrix,

$$
|\Lambda - \lambda I| = \begin{vmatrix}
A-\lambda & \gamma A & \gamma A & E & \epsilon E \\
\gamma A & A-\lambda & \gamma A & E & E \\
\gamma A & \gamma A & A-\lambda & E & E \\
E & E & E & C-\lambda & \omega C \\
\epsilon E & E & E & \omega C & C-\lambda
\end{vmatrix}
$$

$$= -2^{-4}\{-A(1-\gamma)+\lambda\} \times \begin{vmatrix} 0 & 2\{A(1+2\gamma)-\lambda\} & E(1-\epsilon) & E(5+\epsilon) \\ -2\{A(1-\gamma)-\lambda\} & 2\{A(1+\gamma)-\lambda\} & 0 & 4E \\ 2E(\epsilon-1) & 4E & 0 & 2\{C(1+\omega)-\lambda\} \\ 2E(\epsilon-1) & 0 & 2\{-C(1-\omega)+\lambda\} & 0 \end{vmatrix}.$$

We put $r = \frac{X_2}{X_1}$. In addition, C is expressed by A and E as

$$C = \frac{3(X_1+3X_2)E^2}{(3X_1+X_2)A}.$$

Therefore, $|\Lambda - \lambda I|$ becomes

$$|\Lambda - \lambda I| = -2^{-4}\{-A(1-\gamma)+\lambda\} \times \begin{vmatrix} 0 & 2\left(\frac{27+r}{3(3+r)}A-\lambda\right) & \frac{4r}{3+r}E & \frac{2(9+r)}{3+r}E \\ -2\left(\frac{4r}{3(3+r)}A-\lambda\right) & 2\left(\frac{2(9+r)}{3(3+r)}A-\lambda\right) & 0 & 4E \\ 2\frac{-4r}{3+r}E & 4E & 0 & 2\left(\frac{6(1+r)E^2}{(3+r)A}-\lambda\right) \\ 2\frac{-4r}{3+r}E & 0 & -2\left(\frac{12rE^2}{(3+r)A}-\lambda\right) & 0 \end{vmatrix}$$

$$= -2^{-4}\{-A(1-\gamma)+\lambda\} \times \frac{(-128)r^2}{9(3+r)^4}E^4 \times \begin{vmatrix} 0 & (27+r)-4rv\lambda & 1 & 9+r \\ 1-v\lambda & 2(9+r)-4rv\lambda & 0 & 2(3+r) \\ 3 & 6(3+r) & 0 & 6(1+r)-2rz\lambda \\ 3 & 0 & -6+z\lambda & 0 \end{vmatrix}. \qquad (77)$$

From the coefficient of the determinant, the first eigenvalue is obtained as

$$\lambda_1 = A(1-\gamma) = 4X_2R_{1R}^2 < 0.$$

It is proved that the determinant is equal to 0 when $\lambda = 0$ is substituted. Thus, the fourth-order polynomial of λ, Eq. (77), has a factor λ. Thus, by dividing the polynomial by 2λ, we obtain the following cubic equation:

$$4r^2v^2z^2\lambda^3 - rvz(12v+5rz+27z+36rv)\lambda^2 + (27rz^2+240rvz+r^2z^2+72r^2v^2+72rv^2+24r^2vz)\lambda - 432rv - 120rz = 0. \qquad (78)$$

We calculate v and z as

$$v = \frac{3(3+r)}{4rA} = \frac{2}{(\Xi_1^2-\Xi_2^2)X_2}, \quad z = \frac{(3+r)A}{2rE^2} = \frac{12}{(9\Xi_2^2-\Xi_1^2)X_2}.$$

Dividing Eq. (78) by $4r^2v^2z^2$, we obtain

$$f(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where

$$\begin{aligned}
a_2 &= -\frac{1}{8}\Big\{(\Xi_1^2 - \Xi_2^2)(27X_1 + 5X_2) + 2(9\Xi_2^2 - \Xi_1^2)(X_1 + 3X_2)\Big\},\\
a_1 &= \frac{1}{16}\Big\{X_2(\Xi_1^2 - \Xi_2^2)^2(27X_1 + X_2) + 2X_2(9\Xi_2^2 - \Xi_1^2)^2(X_1 + X_2)\\
&\qquad + 4X_2(\Xi_1^2 - \Xi_2^2)(9\Xi_2^2 - \Xi_1^2)(10X_1 + X_2)\Big\},\\
a_0 &= -\frac{1}{4}X_1X_2^2(\Xi_1^2 - \Xi_2^2)(9\Xi_2^2 - \Xi_1^2)(11\Xi_2^2 + \Xi_1^2).
\end{aligned}$$

Since the eigenvalues of a real symmetric matrix are real numbers, the solutions of $f(\lambda) = 0$ should be real numbers. This means that $f(\lambda) = 0$ has three real solutions. Furthermore, $f'(\lambda) = 0$ should have two real solutions. Now, we show that the function $f(\lambda)$ has three negative real solutions.

From the relations $X_l < 0$, $\Xi_1 > \Xi_2$, and $\Xi_1 < 3\Xi_2$, the coefficients a_0, a_1, a_2 are all positive. Let ξ and η ($\xi < \eta$) be two real solutions of $f'(\lambda) = 0$. The conditions that $f(\lambda) = 0$ has three negative real solutions are the following:

$$1.\ f(0) > 0\ ,\ 2.\ \eta < 0.$$

We investigate these conditions.

1. Since $f(0) = a_0$ and $a_0 > 0$, $f(0) > 0$ follows.

2. The first derivative of $f(\lambda)$ becomes

$$f'(\lambda) = 3\lambda^2 + 2a_2\lambda + a_1 = 0. \tag{79}$$

Since it has two real solutions, $a_2^2 - 3a_1 > 0$ follows. Then, the solutions of Eq. (79), ξ and η, are

$$\xi = \frac{-a_2 - \sqrt{a_2^2 - 3a_1}}{3},\quad \eta = \frac{-a_2 + \sqrt{a_2^2 - 3a_1}}{3}.$$

Since $a_1 > 0$ and $a_2 > 0$, $\eta < 0$ follows.

Therefore, Λ has one zero and four negative eigenvalues. Thus, the Hessian matrix in the CA has zero (2-fold) and four positive eigenvalues. Therefore, this implies that the free energy of the CA has the shape of a valley and the CA is stable.

Symmetric mixed solution: $\mathbf{S}_4$

We consider the case of $p = 3$. We assume $R_{1I} = 0$ from the rotational symmetry. In addition, we assume $R_{2I} = R_{3I} = 0$. Then, we obtain $R_1 = R_2 = R_3$. In Appendix D, it is proved that $R_{1R} = R_{2R} = R_{3R}$ is a solution but there is no solution in which one or more of the signs of R_{1R}, R_{2R}, and R_{3R} are reversed. Below, we assume $R_{1R} = R_{2R} = R_{3R} > 0$. The values of u_l , R_{lR}, R_{lI}, and R are

$$\begin{aligned} u_2 &= u_3 = u_4 = \frac{1}{1-a}\Big(\frac{4}{\beta J} - 3(1+3a)u_1\Big), \\ R_{1R} &= \frac{x_1}{3\beta J} = \frac{x_2}{\beta J} = R_{2R} = R_{3R}, \quad R = \frac{x_1}{\sqrt{3}\beta J}. \end{aligned}$$

x_1 is determined by Eq. (D·84). See Appendix D for details.

The values of $c_{\mu\mu}$ and $c_{\mu\nu}$ are

$$c_{\mu\mu} = \frac{3}{\beta J} - 2(1+3a)u_1, \quad c_{\mu\nu} = -\frac{1}{\beta J} + (1+3a)u_1, \qquad (\mu \neq \nu).$$

Then, we obtain

$$\delta_{\mu\nu} - \beta J c_{\mu\nu} = \begin{cases} -2 + 2(1+3a)\beta J u_1, & (\mu = \nu), \\ 1 - (1+3a)\beta J u_1, & (\mu \neq \nu). \end{cases}$$

In the symmetric mixed solution S_4, we have $u_2 = u_3 = u_4, x_1 = 3x_2 = 3x_3 = 3x_4$. The components of the Hessian matrix are calculated as

$$\begin{aligned} \mathcal{H}_{1R1R} &= 2JN\Big(-1 + (1+3a)\beta J u_1 - (\beta J)^3 R_1^2 \{\frac{1+3a}{8}\frac{u'(x_1)}{x_1} 9 + \frac{1-a}{8}\frac{u'(x_2)}{x_2} 3\}\Big) \\ &= \mathcal{H}_{2R2R} \equiv A, \\ \mathcal{H}_{1R2R} &= JN\Big(1 - (1+3a)\beta J u_1 - 2(\beta J)^3 R_1^2 \{\frac{1+3a}{8}\frac{u'(x_1)}{x_1} 9 - \frac{1-a}{8}\frac{u'(x_2)}{x_2} 3\}\Big) \\ &= \mathcal{H}_{1R3R} = \mathcal{H}_{2R3R} \equiv B, \\ \mathcal{H}_{1I1I} &= JN\Big(-2 + 2(1+3a)\beta J u_1\Big) = \mathcal{H}_{2I2I} = \mathcal{H}_{3I3I} \equiv C, \\ \mathcal{H}_{1I2I} &= JN\Big(1 - (1+3a)\beta J u_1\Big) = \mathcal{H}_{1I3I} = \mathcal{H}_{2I3I} \equiv D, \\ \mathcal{H}_{1R1I} &= JN\Big(-(\beta J)^3 \sum_{l=1}^{2^p} P_l \frac{u'(x_l)}{x_l} \eta_l^\mu \eta_l^\nu \zeta_{lR}\zeta_{lI}\Big) = 0 \\ &= \mathcal{H}_{2R2I} = \mathcal{H}_{3R3I} = \mathcal{H}_{1R2I} = \mathcal{H}_{1R3I} = \mathcal{H}_{2R3I}. \end{aligned}$$

Thus, $\mathcal{H}$ is expressed as

$$\mathcal{H} = \begin{array}{c} \\ 1R \\ 2R \\ 3R \\ 1I \\ 2I \\ 3I \end{array} \begin{array}{c} \begin{array}{cccccc} 1R & 2R & 3R & 1I & 2I & 3I \end{array} \\ \begin{pmatrix} A & B & B & 0 & 0 & 0 \\ B & A & B & 0 & 0 & 0 \\ B & B & A & 0 & 0 & 0 \\ 0 & 0 & 0 & C & D & D \\ 0 & 0 & 0 & D & C & D \\ 0 & 0 & 0 & D & D & C \end{pmatrix} \end{array}.$$

The characteristic equation of an $n \times n$ matrix with the diagonal components A and the other components B is

$$\{A - \lambda + (n-1)B\}(A - \lambda - B)^{n-1} = 0.$$

Thus, we obtain the six eigenvalues of $\mathcal{H}$ as

$$\lambda = A + 2B,\ A - B\ \text{(2-fold)},\ C + 2D,\ C - D\ \text{(2-fold)}.$$

Let us study the signs of these eigenvalues. $A + 2B$ becomes

$$\begin{aligned} A + 2B &= 2JN\Big(-1 + (1+3a)\beta J u_1 - \frac{1}{8}(\beta J)^3 R_1^2\{(1+3a)\frac{u'(x_1)}{x_1}9 + (1-a)\frac{u'(x_2)}{x_2}3\} \\ &\quad +1 - (1+3a)\beta J u_1 - \frac{2}{8}(\beta J)^3 R_1^2\{(1+3a)\frac{u'(x_1)}{x_1}9 + (1-a)\frac{u'(x_2)}{x_2}3\}\Big) \\ &= -\frac{9}{4}JN(\beta J)^3 R_1^2\Big((1+3a)\frac{u'(x_1)}{x_1}3 + (1-a)\frac{u'(x_2)}{x_2}\Big). \end{aligned}$$

Since $J > 0, N > 0, x_l > 0, 0 < a \leq 1$, and the function u_l decreases monotonically, i.e., $u_l' < 0$, we obtain $A + 2B > 0$. We find that

$$C + 2D = 2JN\{-1 + (1+3a)\beta J u_1 + 1 - (1+3a)\beta J u_1\} = 0.$$

$A - B = C - D$ is proved as

$$\begin{aligned} A - B &= JN\Big(-2 + 2(1+3a)\beta J u_1 - \frac{2}{8}(\beta J)^3 R_1^2\{(1+3a)\frac{u'(x_1)}{x_1}9 + (1-a)\frac{u'(x_2)}{x_2}3\} \\ &\quad -1 + (1+3a)\beta J u_1 + \frac{2}{8}(\beta J)^3 R_1^2\{(1+3a)\frac{u'(x_1)}{x_1}9 + (1-a)\frac{u'(x_2)}{x_2}3\}\Big) \\ &= 3JN\Big(-1 + (1+3a)\beta J u_1\Big) \\ &= C - D. \end{aligned}$$

Thus, the sign of $A - B$ determines the stability of S_4. That is, if this is positive, the solution is stable. The condition for this is

$$-1 + (1+3a)\beta J u_1 \quad > \quad 0.$$

We define $g(T) = u(x_1(T)) - y(T)$ and $y(T) = \frac{T}{(1+3a)J}$. Then, the above condition is equivalent to

$$g(T) \quad > \quad 0. \tag{80}$$

The critical point for S_4 is $T_{\rm c}^{(S_4)} = \frac{(1+2a)J}{2}$. Thus, we obtain

$$y(T_{\rm c}^{(S_4)}) = \frac{1+2a}{2(1+3a)} < \frac{1}{2}.$$

Since $x_1(\ T_{\rm c}^{(S_4)})$=0, we obtain $u(x_1(\ T_{\rm c}^{(S_4)})$=1/2. Therefore, we obtain $g(\ T_{\rm c}^{(S_4)}) > 0$. $x_1(T)$ is determined by the following equation [Eq. (54)]:

$$\frac{T}{J} \quad = \quad \frac{3}{4}(1+3a)u(x_1(T)) + \frac{1}{4}(1-a)u(\frac{x_1(T)}{3}). \tag{81}$$

The derivative $x_1'(T)$ is calculated as

$$x_1'(T) = \frac{12}{\{9(1+3a)u'(x_1(T)) + (1-a)u'(\frac{x_1(T)}{3})\}J}.$$

Since $u' < 0$, we obtain $x_1'(T) < 0$. The derivative $g'(T)$ is

$$\begin{aligned} g'(T) &= u'(x_1(T))x_1'(T) - y'(T) \\ &= \frac{12}{\{9(1+3a) + (1-a)\frac{u'(\frac{x_1(T)}{3})}{u'(x_1(T))}\}J} - \frac{1}{(1+3a)J}. \end{aligned}$$

Let us consider the limit $T \to 0$. As $T \to 0$, L.H.S. of Eq. (81) $\to 0$, and this implies that $u(x_1(T)) \to 0$ as $T \to 0$. Thus, $g(+0) = 0$. Since we have $x_1(T) \gg 1$ when $T \sim 0$, we obtain $u(x) \sim \frac{1}{x}$. The derivative $u'(x)$ is estimated for $x \gg 1$ as

$$u'(x) \quad \simeq \quad -\frac{1}{x^2}.$$

Thus, we obtain for $T \sim 0$

$$u'(x_1(T))x_1'(T) \quad \sim \quad \frac{12}{\{9(1+3a) + (1-a)\frac{x_1^2(T)}{(\frac{x_1}{3})^2}\}J} \sim \frac{2}{3(1+a)J}.$$

Therefore, when $T \to 0$, we have

$$\begin{aligned} g'(+0) &= \frac{2}{3(1+a)J} - \frac{1}{(1+3a)J} \\ &= \frac{3a-1}{3(1+a)(1+3a)J}. \end{aligned} \tag{82}$$

(a) Case of $a < \frac{1}{3}$

In this case, from Eq. (82), $g'(+0) < 0$ follows. Since $g(+0) = 0$, we obtain $g(T) < 0$

for $0 < T \ll 1$. Since $g(T_c^{(S_4)}) > 0$, there is T that satisfies $g(T) = 0$ in $(0, T_c^{(S_4)})$. We write this temperature as $\tilde{T}$. That is, $u(x_1(\tilde{T})) = \frac{\tilde{T}}{(1+3a)J}$ at $\tilde{T}$, and this is simply the equation for u_1 of the CA, see Eq. (48). In addition, for the symmetric mixed solution S_4, from Eqs. (52)-(54), we obtain

$$u_2 = u_3 = u_4 \quad = \quad \frac{1}{1-a}\Big(\frac{4T}{J} - 3(1+3a)u_1\Big). \tag{83}$$

Substituting $T = \tilde{T}$ in Eq. (83), we have

$$u_2(\tilde{T}) \quad = \quad \frac{\tilde{T}}{(1-a)J}. \tag{84}$$

This is the equation to be satisfied for $u_2 = u_3 = u_3$ of the CA, Eq. (49). Moreover, for S_4, we have the condition $x_1 = 3x_2$. Thus, $\tilde{T}$ satisfies the conditions for the critical temperature $T_c^{(CA)}$of the CA. Since $T_c^{(CA)}$is unique, we obtain $\tilde{T} = T_c^{(CA)}$. Thus, we obtain $g(T) < 0$ for $T <T_c^{(CA)}$and $g(T) > 0$ for $T >T_c^{(CA)}$. Therefore, S_4 is stable for $T_c^{(CA)}< T < T_c^{(S_4)}$. Furthermore, we find that S_4 and the CA do not coexist. S_4 is stabilized when the CA ceases to exist.

(b) Case of $a > \frac{1}{3}$

In this case, since $g'(+0) > 0$ and $g(+0) = 0$, we obtain $g(T) > 0$ for $0 < T \ll 1$. As discussed in the above case, if $g(T) = 0$, this temperature is the critical temperature of the CA. However, for $a > \frac{1}{3}$, the CA does not exist. Therefore, $g(T) \neq 0$ for $0 < T \leq T_c^{(S_4)}$. Since $g(T_c^{(S_4)}) > 0$, $g > 0$ holds for $T \leq T_c^{(S_4)}$. Thus, the solution S_4 is always stable as long as it exists.

In Fig. 4, we show the graph of the functions $u(x_1(T))$ and $y(T)$ in case (a).

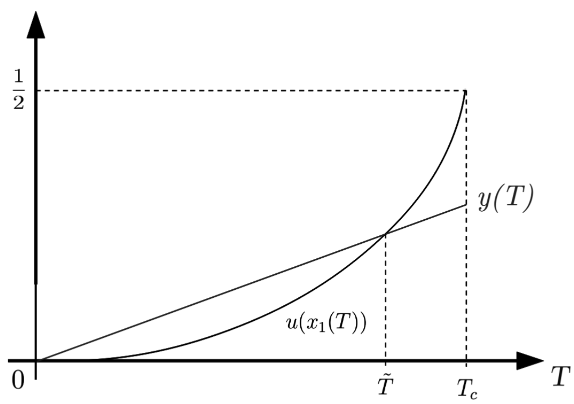

Fig. 4. Functions $u(x_1(T))$ and $y(T)$.

5. Numerical results

We perform MCMCs. We set $J = 1$ in all simulations.

5.1 Phase diagram in (a, T) plane

In Fig. 3, we displayed the phase diagram in the (a, T) plane. We performed MCMCs with $N = 20000$. The numerical method used to obtain stationary states is as follows. As an initial condition, we take ξ^1, and add a perturbation $-h \sum_{j=1}^{N} \cos(\phi_j - \phi_j^1)$ with $h = 0.005$ to the Hamiltonian H in Eq. (1). Here, ϕ_j^μ is defined by $\xi_j^\mu = e^{i\phi_j^\mu}$ for $\mu = 1, 2, \cdots, p$. After the system settles to a stationary state, we identify the state as follows:

Para: $|R_1 - R_2| < 0.02$, $|R_1 - R_3| < 0.02$, and $R_1 < 0.05$.

S_4: $|R_1 - R_2| < 0.02$, $|R_1 - R_3| < 0.02$, and $R_1 > 0.05$.

CA: R_1 is greater than R_2 by more than 0.02. In order to confirm that the final state obtained numerically is really the CA state, we change the perturbation to a new perturbation, $-h \sum_{j=1}^{N} \cos(\phi_j - \phi_j^2)$ with $h = 0.005$, add it to the final state, and check that the new final state satisfies $R_2 - R_1 > 0.02$.

As seen from Fig. 3, the theoretical and numerical results agree reasonably well.

5.2 Temperature dependences of order parameters

First of all, we show theoretical and numerical results of the temperature dependence of the order parameter R in Fig. 5 for $a = 0$ and in Fig. 6 for $a = 0.1$. In the numerical simulations, N is set to 10^4, and the total number of Monte Carlo sweeps (MC sweeps) is 10^4. Here, one MC sweep corresponds to N updates of the XY spins. We took the average during the last 5000 MC sweeps. Furthermore, we took the sample average over 50 samples. We display the average and the standard deviation for R, but the latter is too small to observe. The theoretical and numerical results agree reasonably well.

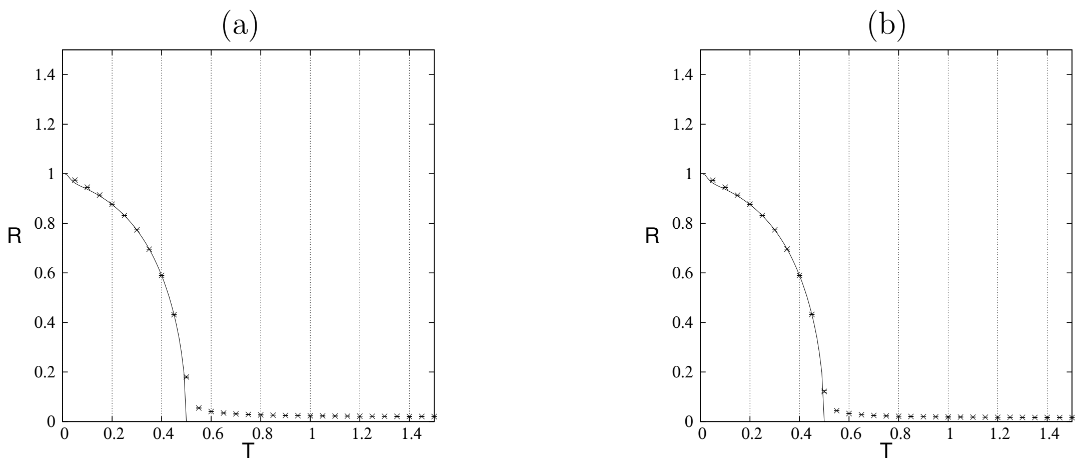

Fig. 5. Temperature dependences of R for $a = 0$. Solid curve: theoretical results of the CA. Symbols: simulation results with error bars. (a) $p = 2$, (b) $p = 3$.

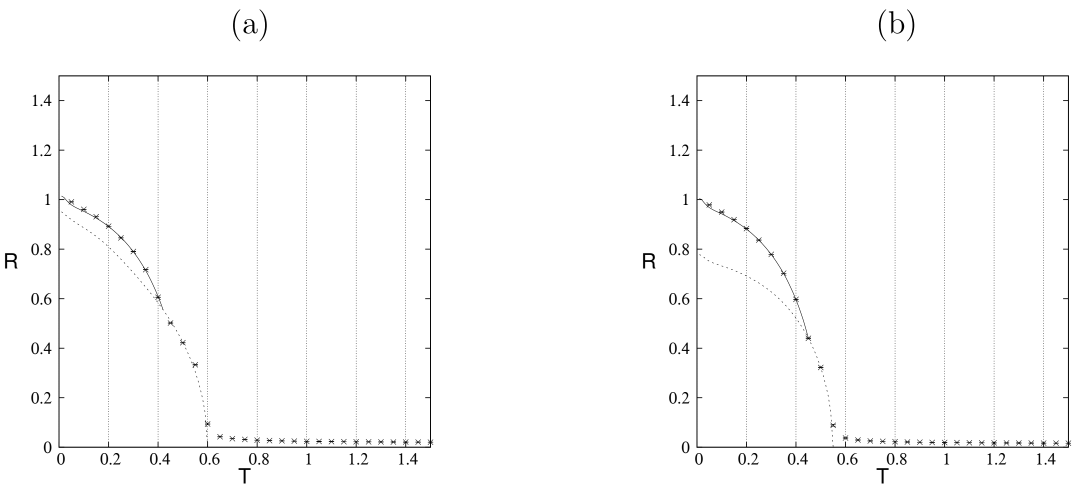

Fig. 6. Temperature dependences of R for $a = 0.1$. Curves: theoretical results. Solid curve: CA. Symbols: simulation results with error bars. (a) $p = 2$. Dotted curve: S_1. (b) $p = 3$. Dotted curve: S_4.

5.3 Maximum number of patterns for which the CA exists

Next, we study the maximum number of patterns p_c for which the CA exists. Theoretically, as long as the self-averaging property holds, p_c can take any value for $a = 0$, whereas $p_c = 3$ for $0 < a < \frac{1}{3}$ and $p_c = 2$ for $a > \frac{1}{3}$.

We perform MCMCs for $N = 4000$ and 8000 and $T = 0.1$. We draw R_μ from 0 to 20000 MC sweeps at intervals of 100 MC sweeps. We set the initial configuration as the CA in order to reduce the time to reach the CA when it exists. We used the following criterion to judge whether the resultant solution is the CA. From 10^4 to 20000 MC sweeps, at every MC sweep we selected the largest and second largest values of $\{R_\mu\}$, say, $R^{1\text{st}}$ and $R^{2\text{nd}}$. We defined $\Delta R = R^{1\text{st}} - R^{2\text{nd}}$ and calculated the standard deviation of ΔR, σ_R. We took 10 samples, and obtained 10 σ_R. We selected the largest one among the σ_R, say, $\sigma_R^{\max}$. If $\sigma_R^{\max}$ exceeded some value, σ^*, we judged that the CA exists. Empirically, $\sigma^* = 0.1$ gave reasonable results.

<u>Case of $a = 0$</u>

We show the numerical results in Fig. 7 for $N = 8000$. It seems that the CA exists until $p = 36$. Let us study the condition for the existence of the CA for finite N. In finite-size

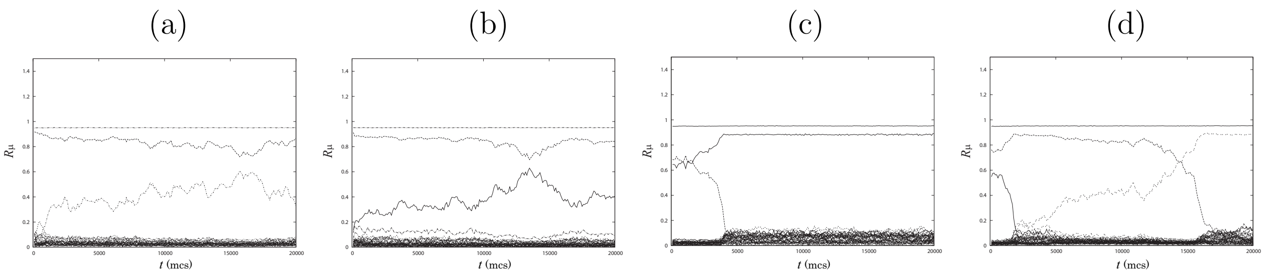

Fig. 7. Time series of R_μs. $a = 0, N = 8000$. mcs denotes Monte Carlo sweeps. (a) $p = 25$, (b) $p = 31$, (c) $p = 35$, (d) $p = 36$.

systems, in order that the self-averaging property holds, $2^p < N$ should be satisfied. Thus, the critical p_c for the number of spins N is estimated from $2^{p_c} \sim N$. Thus, $p_c \approx \frac{1}{\ln 2} \ln N$. When $N = 4000$ and 8000, $\frac{1}{\ln 2} \ln N \simeq 12$ and 13, respectively. These estimates are consistent with the numerical results of $p_c \sim 20$ and 30, respectively.

<u>Case of $a > 0$</u>

We perform MCMCs for $a = 0.1$. Numerical results are shown in Fig. 8 for $N = 8000$. Note that the CA exists only for $p = 2$ and 3 as the theory predicts.

5.4 Addition of noise to patterns

When $a > 0$, we theoretically and numerically found that the CA exists only for $p = 2$ and 3, although when $a = 0$, p_c can take any value as long as the self-averaging property holds theoretically, and $p_c \sim \ln N$ numerically. In realistic situations, there is external noise. Therefore, we study the case that patterns are subject to external noise when $a > 0$. It is expected that we can produce similar situations to the case of $a = 0$ and make the CA reappear by the addition of noise because noise reduces the

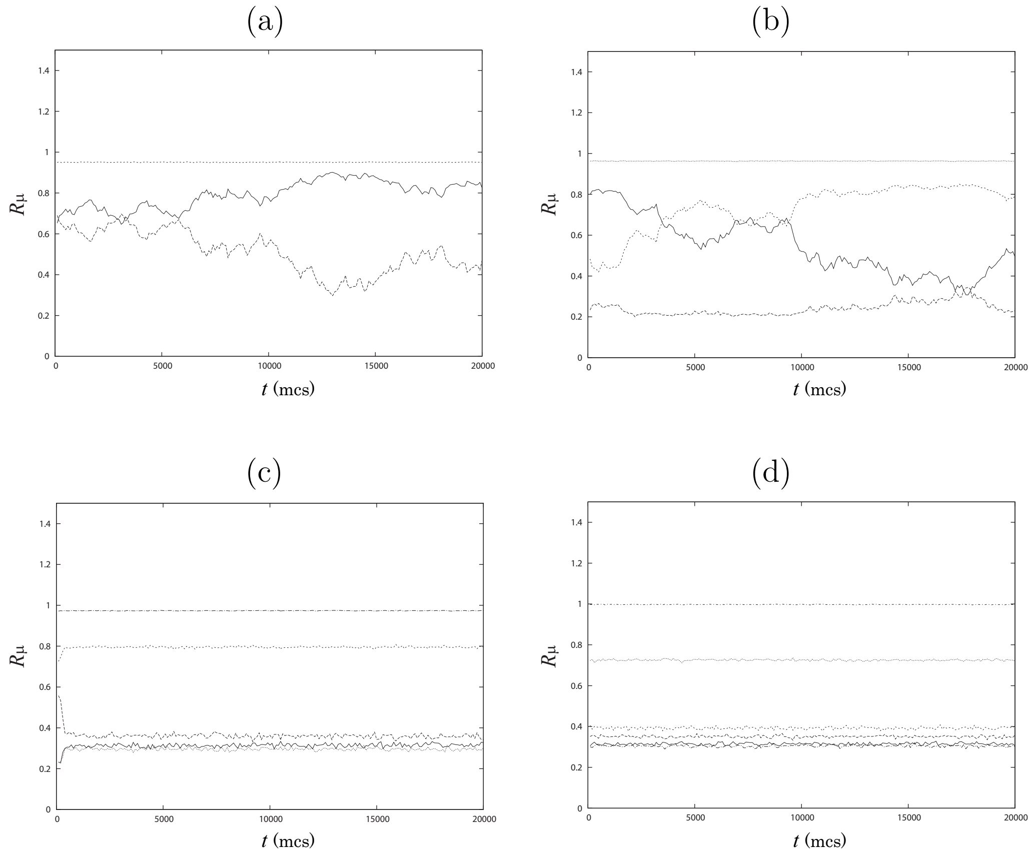

Fig. 8. Time series of R_μ. $a = 0.1, N = 8000$. (a) $p = 2$, (b) $p = 3$, (c) $p = 4$, (d) $p = 5$.

correlation among patterns.

Noise is introduced in such a way that the sign of each pattern ξ_i^μ is reversed with some probability, say λ. Then, for $0 < \lambda \leq 1$, the substantial correlation a' between any two patterns becomes $a' = (1 - 2\lambda)^2 a$ for $\lambda \leq \frac{1}{2}$ and $a' = -(1 - 2\lambda)^2 a$ for $\lambda > \frac{1}{2}$. Thus, as $\lambda \to \frac{1}{2}$, $a' \to 0$. Fixing $a = 0.1$ and $T = 0.1$, we performed MCMCs for $N = 8000$ and for several values of p and λ. We set the initial configuration at random.

We took 10 samples, calculated the standard deviation σ_R, and determined the maximum of σ_R, $\sigma_R^{\max}$, as before. We show the time series of R_μ for the sample with $\sigma_R^{\max}$ in Figs. 9-11. We find that p_c increases from 3 as λ increases as expected. For example, p_c is 4, 4, and 6 for $\lambda = 0.2, 0.25$, and 0.3, respectively.

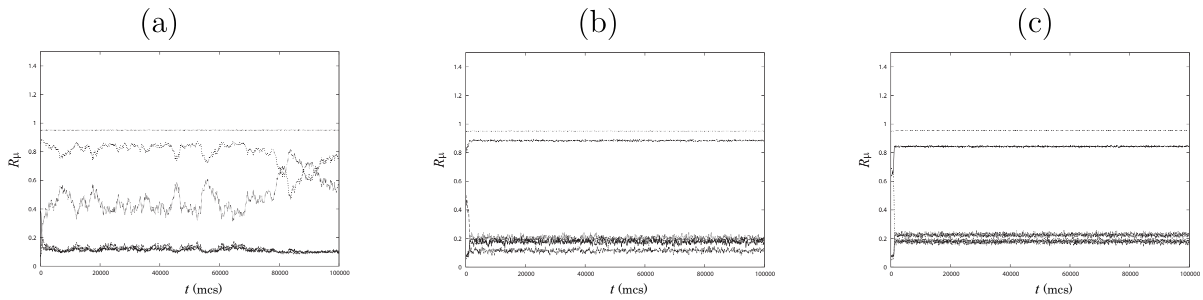

Fig. 9. Time series of R_μ. $\lambda = 0.2, N = 8000$. (a) $p = 4$, (b) $p = 5$, (c) $p = 6$.

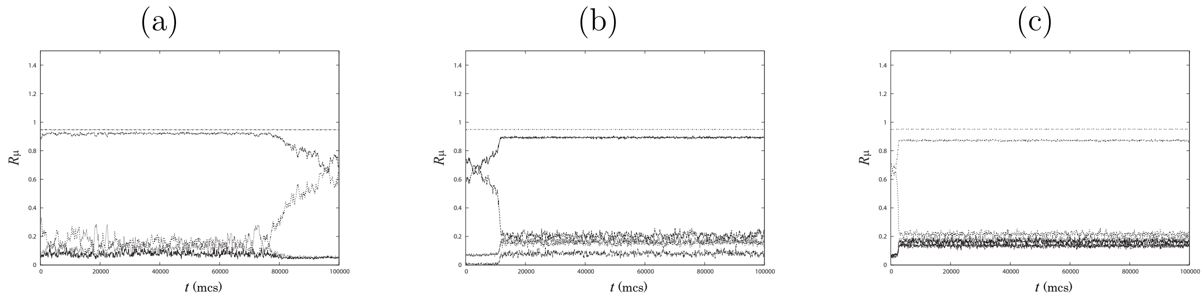

Fig. 10. Time series of R_μ. $\lambda = 0.25, N = 8000$. (a) $p = 4$, (b) $p = 5$, (c) $p = 6$.

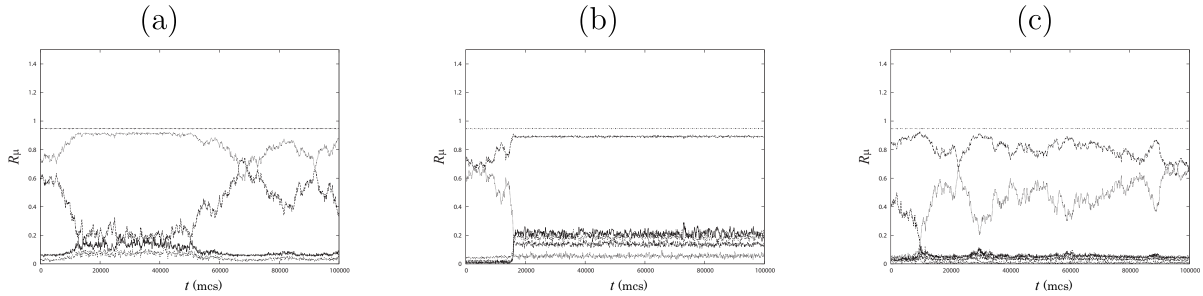

Fig. 11. Time series of R_μ. $\lambda = 0.3, N = 8000$. (a) $p = 4$, (b) $p = 5$, (c) $p = 6$.

6. Summary and Discussion

We have analyzed the classical XY model with the associative-memory-type interaction for the case that $N \gg 1$ and the self-averaging property holds, with and without the correlation a between any two patterns.

Firstly, we summarize the theoretical results. In Table III, we list the stable solutions.

	$a = 0$	$a > 0$
$p = 2$	Continuous attractor ($T < \frac{J}{2}$) Memory pattern ($T < \frac{J}{2}$)	Continuous attractor ($T < \frac{(1-a)J}{2}$) Symmetric mixed solution S_1 ($\frac{(1-a)J}{2} < T < \frac{(1+a)J}{2}$)
$p = 3$	Continuous attractor ($T < \frac{J}{2}$) Memory pattern ($T < \frac{J}{2}$)	Continuous attractor ($T < T_{\mathrm{c}}^{(\mathrm{CA})}$) Symmetric mixed solution S_4 ($T_{\mathrm{c}}^{(\mathrm{CA})} < T < \frac{(1+2a)J}{2}$)

Table III: Stable solutions for $p = 2$ and 3.

For general p, we studied the condition for the existence of the CA. When $a = 0$, the CA exists for any p and is stable as long as it exists. Among the overlaps with memory patterns, $\{R_\mu\}$, only two are nonzero. The critical temperature is $T_{\mathrm{c}}^{(\mathrm{CA})} = \frac{J}{2}$ for any p. Since memory patterns are located at both ends of the CA, their stabilities are the same as that of the CA. On the other hand, when $a > 0$, the CA exists only when $p = 2$ and 3. The reason for this is that the number of conditions becomes larger than the number of independent variables for $p \geq 4$. The CA exists and is stable for $T < T_{\mathrm{c}}^{(\mathrm{CA})} (= \frac{(1-a)J}{2})$ when $p = 2$. The symmetric mixed solution S_1 exists for $T < T_{\mathrm{c}}^{(S_4)} (= \frac{1+a}{2} J)$. It is unstable for $0 < T < T_{\mathrm{c}}^{(\mathrm{CA})}$ and becomes stable when the CA disappears. That is, a coexistence region of the CA and the symmetric mixed solution S_1 does not exist. When $p = 3$, the CA exists and is stable below $T_{\mathrm{c}}^{(\mathrm{CA})}$, which is determined by $x_1 = 3x_2$. A pure memory pattern does not exist when $a \neq 0$, but its modified version appears at both ends of the CA. The symmetric mixed solution S_4 exists for $T < T_{\mathrm{c}}^{(S_4)} (= \frac{1+2a}{2} J)$. It is unstable for $0 < T < T_{\mathrm{c}}^{(\mathrm{CA})}$ and becomes stable when the CA disappears. That is, a coexistence region of the CA and the symmetric mixed solution S_4 does not exist as in the case of $p = 2$. For $p = 2$ and 3 and for both $a = 0$ and $a > 0$, several other solutions exist but all of them are unstable.

Secondly, we summarize the numerical results. We performed MCMCs and calculated the critical number of patterns p_c until which the CA exists. When $a = 0$, the CA exists until $p_c \sim 20$ and ~ 30 for $N = 4000$ and 8000, respectively. Theoretically, the CA exists and is stable for any p as long as the self-averaging property holds. The reason for this disagreement is considered to be due to the breakdown of the self-averaging in the finite size system. We estimated p_c for finite N as $p_c \sim \ln N / \ln 2$ and we found that this is consistent with numerical results. On the other hand, when $a > 0$, the CA exists until $p_c = 3$ for $N = 8000$. This result completely agreed with the theoretical result. Furthermore, for $a > 0$, we added external noise to the components of patterns, because we expected that the correlation between patterns would be weakened by the addition of noise to patterns. By MCMCs, we found that p_c increases from 3 as the probability λ that each component is reversed increases as expected.

Now, let us consider the meaning of the existence of the CA when the present model is regarded as an associative memory model. In real brains, after a memory is retrieved, another memory is sometimes spontaneously retrieved without any stimulation, or when an external stimulus is applied, a memory that is related to the stimulus is retrieved. That is, it seems that many memories in a real brain are “connected” in a sense. Such phenomena do not take place for models that have only point attractors such as models composed of the Ising spins. On the other hand, in the present model, the CA exists between any two embedded patterns. Thus, after a pattern ξ^μ is retrieved, another pattern can be retrieved spontaneously. Moreover, if an external stimulus that lies on a path from pattern ξ^μ to pattern ξ^ν is added, pattern ξ^ν is retrieved. That is, the CA is considered to be able to realize the feature of real brains mentioned above.

Finally, we list several future problems. The first is to examine the system size N dependence of the critical number of patterns p_c for $a = 0$. Extensive theoretical and numerical studies are necessary. The second is the theoretical analysis of the effects of adding external noise for $a > 0$ in order to make the CA reappear. The third one is to extend the present study to the case that patterns are divided into clusters in such a way that patterns in any cluster are correlated but those in two different clusters are not correlated.

Acknowledgements

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Appendix A: Derivation of Free Energy and Saddle Point Equations

The associative memory interaction is expressed as

$$J_{ij} = \frac{J}{N}\sum_{\mu=1}^{p}\xi_i^\mu \xi_j^\mu. \tag{A·1}$$

The order parameter is defined as follows:

$$R_{\mu R} = \frac{1}{N}\sum_{i=1}^{N}\xi_i^\mu \cos\phi_i, \tag{A·2}$$

$$R_{\mu I} = \frac{1}{N}\sum_{i=1}^{N}\xi_i^\mu \sin\phi_i. \tag{A·3}$$

The Hamiltonian of the classical XY model is

$$H = -\sum_{i<j} J_{ij}\boldsymbol{X}_i \cdot \boldsymbol{X}_j \tag{A·4}$$

$$= -\frac{NJ}{2}\sum_{\mu=1}^{p}\{(R_{\mu R})^2 + (R_{\mu I})^2\} + \frac{Jp}{2}. \tag{A·5}$$

In order to analyze the XY model by the method of statistical mechanics, we introduce the temperature T and calculate the partition function Z. We put $k_B = 1$, so $\beta = \frac{1}{T}$. The partition function Z is expressed as

$$Z = \int_0^{2\pi} d\Phi \mathrm{e}^{\frac{N\beta J}{2}\sum_{\mu=1}^{p}\{(R_{\mu R})^2+(R_{\mu I})^2\}-\frac{\beta Jp}{2}}, \tag{A·6}$$

where $\int_0^{2\pi} d\Phi = \int_0^{2\pi} d\phi_1 \cdots \int_0^{2\pi} d\phi_N$. By the HubbardStratonovich−transformation, we obtain

$$Z = \quad \mathrm{e}^{-\frac{\beta J}{2}p} \int_0^{2\pi} d\Phi \int dy_c^1 \cdots dy_c^p dy_s^1 \cdots dy_s^p \Big(\sqrt{\tfrac{N\beta J}{2\pi}}\Big)^{2p} e^{\hat{H}}, \tag{A·7}$$

where we define

$$\hat{H} \quad = \quad -\frac{N\beta J}{2} \sum_{\mu=1}^{p} \Big((y_c^\mu)^2 + (y_s^\mu)^2 \Big) + \beta J \sum_{\mu=1}^{p} \Big(y_c^\mu \sum_{j=1}^{N} \xi_j^\mu \cos\phi_j + y_s^\mu \sum_{j=1}^{N} \xi_j^\mu \sin\phi_j \Big). \tag{A·8}$$

By performing integration with respect to $\phi_1, \cdots, \phi_N$, we obtain

$$Z \quad = \quad C \int dy_c^1 \cdots dy_c^p dy_s^1 \cdots dy_s^p \mathrm{e}^{Nf},$$

$$Nf \quad = \quad \ln \int_0^{2\pi} d\Phi e^{\hat{H}} \tag{A·9}$$

$$\quad = \quad -\frac{N\beta J}{2} \sum_{\mu=1}^{N} \{ (y_c^\mu)^2 + (y_s^\mu)^2 \} + \sum_{j=1}^{N} \ln(2\pi I_0(\beta J \Xi_j)), \tag{A·10}$$

$$\Xi_j = \sqrt{(\sum_{\mu=1}^{p} \xi_j^\mu y_c^\mu)^2 + (\sum_{\mu=1}^{p} \xi_j^\mu y_s^\mu)^2}, \tag{A·11}$$

where the constant $C = \Big(\sqrt{\frac{N\beta J}{2\pi}}\Big)^{2p} \mathrm{e}^{-\frac{\beta J}{2}p}$ is of order 1 and Nf is of order N. Since we consider the case $N \gg 1$, we evaluate Z by the saddle point method.

$$Z \simeq C\mathrm{e}^{Nf((y_c^1)^*(y_c^2)^* \cdots (y_c^p)^*(y_s^1)^* \cdots (y_s^p)^*)} = C\mathrm{e}^{Nf^*}$$

Here, $(y_c^\mu)^*$, $(y_s^\mu)^*$ is the saddle point of f, and f^* is the value of f at the saddle point. Therefore, the free energy becomes

$$F \quad = \quad -\frac{1}{\beta} \ln Z \simeq -\frac{1}{\beta} Nf^*.$$

By using (A·9), we calculate $\frac{\partial f}{\partial y_c^\mu} = 0$ and $\frac{\partial f}{\partial y_s^\mu} = 0$ as

$$(y_c^\mu)^* \quad = \quad \langle R_{\mu R} \rangle = \frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu \langle \cos\phi_i \rangle, \tag{A·12}$$

$$(y_s^\mu)^* \quad = \quad \langle R_{\mu R} \rangle = \frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu \langle \sin\phi_i \rangle, \tag{A·13}$$

where $\langle A\rangle = \frac{\int_0^{2\pi} e^{\hat{H}} A d\Phi}{\int_0^{2\pi} e^{\hat{H}} d\Phi}$. By performing the integration, we obtain

$$\langle R_{\mu R}\rangle = \frac{1}{N}\sum_{j=1}^{N}\sum_{\nu=1}^{p}\frac{I_1(\beta J\Xi_j)}{I_0(\beta J\Xi_j)}\xi_j^\mu\xi_j^\nu\frac{1}{\Xi_j}(y_c^\nu)^*, \tag{A·14}$$

$$\langle R_{\mu I}\rangle = \frac{1}{N}\sum_{j=1}^{N}\sum_{\nu=1}^{p}\frac{I_1(\beta J\Xi_j)}{I_0(\beta J\Xi_j)}\xi_j^\mu\xi_j^\nu\frac{1}{\Xi_j}(y_s^\nu)^*. \tag{A·15}$$

Hereafter, we write $\langle R_{\mu R}\rangle$ and $\langle R_{\mu I}\rangle$ as $R_{\mu R}$ and $R_{\mu I}$, respectively, for simplicity. Then, the SPEs are

$$R_{\mu R} = \frac{1}{N}\sum_{j=1}^{N}\sum_{\nu=1}^{p}\frac{I_1(\beta J\Xi_j)}{I_0(\beta J\Xi_j)}\xi_j^\mu\xi_j^\nu\frac{1}{\Xi_j}R_{\nu R}, \tag{A·16}$$

$$R_{\mu I} = \frac{1}{N}\sum_{j=1}^{N}\sum_{\nu=1}^{p}\frac{I_1(\beta J\Xi_j)}{I_0(\beta J\Xi_j)}\xi_j^\mu\xi_j^\nu\frac{1}{\Xi_j}R_{\nu I}. \tag{A·17}$$

From Eq. (A·10), the free energy is

$$F = \frac{NJ}{2}R^2 - \frac{1}{\beta}\sum_{j=1}^{N}\ln(2\pi I_0(\beta J\Xi_j)),$$

where

$$\Xi_j = \sqrt{(\sum_{\mu=1}^{p}\xi_j^\mu R_{\mu R})^2 + (\sum_{\mu=1}^{p}\xi_j^\mu R_{\mu I})^2}.$$

Now, we define the average of all $\{\xi_j^\mu\}$ as $\left[A(\{\xi^\mu\})\right]$. By the self-averaging property, we obtain

$$\frac{1}{N}\sum_{j=1}^{N}A(\{\xi_j^\mu\}) = \left[A(\{\xi^\mu\})\right].$$

Then, the free energy and SPEs are rewritten as

$$F = \frac{NJ}{2}R^2 - \frac{N}{\beta}\ln(2\pi I_0(\beta J\Xi_j))],, \tag{A·18}$$

$$R_{\mu R} = \beta J\sum_{\nu=1}^{p}c_{\mu\nu}R_{\nu R},, \tag{A·19}$$

$$R_{\mu I} = \beta J\sum_{\nu=1}^{p}c_{\mu\nu}R_{\nu I},, \tag{A·20}$$

$$c_{\mu\nu} = \left[u(x_j)\xi_j^\mu\xi_j^\nu\right], \tag{A·21}$$

where $x_j = \beta J\Xi_j$ and $u(x_j) = \frac{I_1(x_j)}{xI_0(x_j)}$.

Appendix B: Properties of the Function $u(x)$

We describe the properties of $u(x) = \frac{I_1(x)}{xI_0(x)}$. The modified Bessel function of the first kind $I_\nu(z)$ is defined for the complex number z and the real number ν, which is an analytic function of z, and when z is real, the function is real. We use the following formula for $I_\nu(z)$:[9)]

$$\left(\frac{d}{zdz}\right)^n (z^{-\nu} I_\nu(z)) = z^{-\nu-n} I_{\nu+n}(z). \tag{B·1}$$

$$I_\mu(z)I_\nu(z) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} I_{\mu+\nu}(2z\cos\theta)\cos\{(\mu-\nu)\theta\}d\theta. \tag{B·2}$$

$$\mathrm{Re}(\mu+\nu) > -1.$$

When ν is an integer n, $I_n(x)$ is expressed as follows:

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} \mathrm{e}^{x\cos\phi}\cos(n\phi)d\phi.$$

In this case, $I_n(x) > 0$ for $x > 0$, $I_0(0) = 1$, and $I_n(0) = 0$ $(n > 0)$. $u(x) \equiv \frac{I_1(x)}{xI_0(x)}$ is C^∞ for any real value x, and $u(0) = \frac{1}{2}$ follows. We put $n = 1$, $\nu = 1$, and $z = x$ in Eq. (B·1) and obtain

$$\frac{d}{dx}(x^{-1}I_1(x)) = x^{-1}I_2(x).$$

Thus,

$$\frac{d}{dx}u(x) = \frac{1}{xI_0(x)^2}(I_2(x)I_0(x) - I_1(x)^2). \tag{B·3}$$

Subtracting Eq. (B·2) with $\mu = 1, \nu = 1$, and $z = x > 0$ from that with $\mu = 2$, $\nu = 0$, and $z = x > 0$, we obtain

$$I_2(x)I_0(x) - I_1(x)^2 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} I_2(2x\cos\theta)\big\{\cos(2\theta) - 1\big\}d\theta.$$

For $x > 0$, $2x\cos\theta$ is greater than or equal to zero in the range of integration. Then $I_2(2x\cos\theta) \geq 0$, and the integral is negative. Thus, $u'(x) < 0$ for $x > 0$. By the saddle point method, the asymptotic form for $x \gg 1$ is

$$I_n(x) \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{x(1-\frac{\phi^2}{2})}d\phi = \mathrm{e}^x\frac{1}{\sqrt{2\pi x}}. \tag{B·4}$$

Therefore, for $x \gg 1$ we obtain

$$u(x) \simeq \frac{1}{x}.$$

When $x \to \infty$, $u(x) \to 0$.

Appendix C: Proof of Relations for $\{\eta_l^\mu\}$

We denote the value of ξ_i^μ in the lth sublattice as $\eta_l^{\mu,(p)}$ when the number of patterns is p (≥ 2). Firstly, we summarize the relations among $\eta_l^{\mu,(p)}$:

$$\eta_l^{1,(p)} = 1, \qquad (p \geq 2, l = 1, \cdots, 2^{p-1}), \tag{C·1}$$

$$\eta_{l+2^{p-1}}^{\mu,(p)} = -\eta_l^{\mu,(p)}, \qquad (p \geq 2, l = 1, \cdots, 2^{p-1}, \mu = 1, \cdots, p), \tag{C·2}$$

$$\eta_l^{\mu,(p+1)} = \eta_l^{\mu-1,(p)}, \qquad (p \geq 2, l = 1, \cdots, 2^p, \mu = 2, \cdots, p). \tag{C·3}$$

We show these relations in the cases of $p = 2$ and 3 in Fig. C·1.

	$\eta_l^{1,(2)}$	$\eta_l^{2,(2)}$
l = 1	1	1
l = 2	1	− 1
l = 3	− 1	− 1
l = 4	− 1	1

	$\eta_l^{1,(3)}$	$\eta_l^{2,(3)}$	$\eta_l^{3,(3)}$
l = 1	1	1	1
l = 2	1	1	− 1
l = 3	1	− 1	− 1
l = 4	1	− 1	1
l = 5	− 1	− 1	− 1
l = 6	− 1	− 1	1
l = 7	− 1	1	1
l = 8	− 1	1	− 1

Fig. C·1. Relations among $\eta_l^{\mu,(2)}$ and $\eta_l^{\mu,(3)}$.

The following relation is derived from Eq. (C·2):

$$\sum_{l=1}^{2^p} \eta_l^{1,(p)} = 0. \tag{C·4}$$

Let us prove the following:

$$\sum_{l=1}^{2^{p-1}} \eta_l^{\mu,(p)} = 0, \qquad (p = 2, 3, \cdots), \qquad (\mu = 2, 3, \cdots, p). \tag{C·5}$$

In the case of $p = 2$, this is obvious from Fig. C·1. For general $p \geq 3$ and $\mu \neq 1$, the left-hand side of Eq. (C·5) becomes

$$\sum_{l=1}^{2^{p-1}} \eta_l^{\mu,(p)} = \sum_{l=1}^{2^{p-1}} \eta_l^{\mu-1,(p-1)} = \sum_{l=1}^{2^{p-2}} \eta_l^{\mu-1,(p-1)} + \sum_{l=1}^{2^{p-2}} \eta_{l+2^{p-2}}^{\mu-1,(p-1)}. \tag{C·6}$$

From Eq. (C·2), it becomes zero. By using these relations, we prove the equations used in the main text by the inductive method.

Proof of Eq. (33)

From Eq. (C·2), Eq. (33) can be written as

$$\sum_{l=1}^{2^p} \eta_l^{\mu,(p)} \eta_l^{\nu,(p)} = 2^p \delta_{\mu\nu}. \tag{C·7}$$

When $\mu = \nu$, this is trivial. Thus, let us study the case of $\nu \neq \nu$.

(i) Case of $p = 2$

From Fig. C·1, we obtain

$$\text{L.H.S.} \quad = \quad \sum_{l=1}^{4} \eta_l^{\mu,(2)} \eta_l^{\nu,(2)} = \eta_1^{\mu,(2)} \eta_1^{\nu,(2)} + \eta_2^{\mu,(2)} \eta_2^{\nu,(2)} + \eta_3^{\mu,(2)} \eta_3^{\nu,(2)} + \eta_4^{\mu,(2)} \eta_4^{\nu,(2)} = 0. \tag{C·8}$$

Thus, Eq. (C·7) is proved.

(ii) Case of $p = m$ (≥ 2)

We assume the following:

$$\sum_{l=1}^{2^m} \eta_l^{\mu,(m)} \eta_l^{\nu,(m)} = 0, \ (\mu \neq \nu). \tag{C·9}$$

For $p = m + 1$, let us prove the following:

$$\sum_{l=1}^{2^{m+1}} \eta_l^{\mu,(m+1)} \eta_l^{\nu,(m+1)} = 0, \ (\mu \neq \nu). \tag{C·10}$$

It is necessary to consider the case that μ or ν is equal to 1 and the case that μ and ν are not equal to 1.

(ii)-(a) The case that μ and ν are not equal to 1

$\eta_l^{\tau,(m+1)}$ ($\tau \neq 1$, $l = 1, \cdots, 2^m$) is equal to $\eta_l^{\tau-1,(m)}$. Thus, by Eqs. (C·2), (C·3), and (C·9), we have

$$\text{L.H.S. of Eq. (C·10)} \quad = \quad 2 \sum_{l=1}^{2^m} \eta_l^{\mu,(m+1)} \eta_l^{\nu,(m+1)} = 2 \sum_{l=1}^{2^m} \eta_l^{\mu-1,(m)} \eta_l^{\nu-1,(m)} = 0.$$

(ii)-(b) The case that μ or ν is equal to 1

We assume $\mu = 1$ without loss of generality. By definition, we have

$$\eta_l^{1,(m+1)} = -\eta_{l+2^m}^{1,(m+1)} = 1, \quad (l = 1, \cdots, 2^m).$$

Since $\nu > 1$, from Eqs. (C·2) and (C·5), we have

$$\text{L.H.S. of Eq. (C·10)} \quad = \quad 2 \sum_{l=1}^{2^m} \eta_l^{1,(m+1)} \eta_l^{\nu,(m+1)} = 1 \times 2 \sum_{l=1}^{2^m} \eta_l^{\nu,(m+1)} = 0.$$

This completes the proof.

Proof of Eq. (70)

Let us prove Eq. (70),

$$\sum_{l=1}^{2^{p-1}} \eta_l^{\mu}\eta_l^{\nu}\eta_l^{1}\eta_l^{2} = \begin{cases} 2^{p-1}, & (\mu,\nu)=(1,2) \text{ or } (2,1), \\ 0, & \text{other cases.} \end{cases}$$

From Eq. (C·2), this is also expressed as follows:

$$\sum_{l=1}^{2^{p}} \eta_l^{\mu}\eta_l^{\nu}\eta_l^{1}\eta_l^{2} = \begin{cases} 2^{p}, & (\mu,\nu)=(1,2) \text{ or } (2,1), \\ 0, & \text{other cases.} \end{cases} \qquad (\text{C}\cdot 11)$$

When $\mu=\nu$, this holds from Eq. (C·7). We next prove Eq. (C·11).

(i) Case of $p=2$

From Fig. C·1,

$$\begin{aligned} \text{L.H.S. of Eq. (C·11)} &= \sum_{l=1}^{4} \eta_l^{\mu,(2)}\eta_l^{\nu,(2)}\eta_l^{1,(2)}\eta_l^{2,(2)} \\ &= \eta_1^{\mu,(2)}\eta_1^{\nu,(2)}\eta_1^{1,(2)}\eta_1^{2,(2)} + \eta_2^{\mu,(2)}\eta_2^{\nu,(2)}\eta_2^{1,(2)}\eta_2^{2,(2)} \\ &\quad + \eta_3^{\mu,(2)}\eta_3^{\nu,(2)}\eta_3^{1,(2)}\eta_3^{2,(2)} + \eta_4^{\mu,(2)}\eta_4^{\nu,(2)}\eta_4^{1,(2)}\eta_4^{2,(2)} \\ &= \begin{cases} 4 & (\mu,\nu)=(1,2) \text{ or } (2,1) \\ 0 & \mu=\nu \end{cases} \\ &= \text{R.H.S. of Eq. (C·11).} \end{aligned}$$

Therefore, Eq. (C·11) holds.

(ii) Case of $p=m$ (≥ 2)

We assume that Eq. (C·11) is true,

$$\sum_{l=1}^{2^{m}} \eta_l^{\mu,(m)}\eta_l^{\nu,(m)}\eta_l^{1,(m)}\eta_l^{2,(m)} = \begin{cases} 2^{m}, & (\mu,\nu)=(1,2) \text{ or } (2,1), \\ 0, & \text{other cases.} \end{cases}$$

Let us prove the following:

$$\sum_{l=1}^{2^{m+1}} \eta_l^{\mu,(m+1)}\eta_l^{\nu,(m+1)}\eta_l^{1,(m+1)}\eta_l^{2,(m+1)} = \begin{cases} 2^{m+1}, & (\mu,\nu)=(1,2) \text{ or } (2,1), \\ 0, & \text{other cases.} \end{cases} \qquad (\text{C}\cdot 12)$$

When $\mu\neq\nu$, it is necessary to consider the case that μ or ν is equal to 1 and the case that μ and ν are not equal to 1.

(ii)-(a) The case that both μ and ν are not equal to 1

The left-hand side of Eq. (C·12) is calculated as

$$\text{L.H.S. of Eq. (C·12)} \quad = \quad 2\sum_{l=1}^{2^m} \eta_l^{\mu,(m+1)}\eta_l^{\nu,(m+1)}\eta_l^{1,(m+1)}\eta_l^{2,(m+1)}.$$

For $l \leq 2^m$, using $\eta_l^{1,(m+1)} = 1$ and $\eta_l^{\tau,(m+1)} = \eta_l^{\tau,(m)}$ for $\tau \geq 2$, it is rewritten as

$$2\sum_{l=1}^{2^m} \eta_l^{\mu-1,(m)}\eta_l^{\nu-1,(m)}\eta_l^{1,(m)}.$$

Furthermore, we decompose the sum using Eq. (C·2),

$$= \quad 2\Big[\sum_{l=1}^{2^{m-1}} \eta_l^{\mu-1,(m)}\eta_l^{\nu-1,(m)}\eta_l^{1,(m)} + \sum_{l=2^{m-1}+1}^{2^m} \eta_l^{\mu-1,(m)}\eta_l^{\nu-1,(m)}\eta_l^{1,(m)}\Big]$$

$$= \quad 2\Big[\sum_{l=1}^{2^{m-1}} \eta_l^{\mu-1,(m)}\eta_l^{\nu-1,(m)}\eta_l^{1,(m)} - \sum_{l=1}^{2^{m-1}} \eta_l^{\mu-1,(m)}\eta_l^{\nu-1,(m)}\eta_l^{1,(m)}\Big] = 0.$$

In the present case, the R.H.S. of Eq. (C·12) is zero and Eq. (C·12) holds.

(ii)-(b) The case that μ or ν is equal to 1

We assume $\mu = 1$ without loss of generality,

$$\begin{aligned} \text{L.H.S. of Eq. (C·12)} \quad &= \quad \sum_{l=1}^{2^{m+1}} \eta_l^{1,(m+1)}\eta_l^{\nu,(m+1)}\eta_l^{1,(m+1)}\eta_l^{2,(m+1)} \\ &= \quad 1 \times \sum_{l=1}^{2^{m+1}} \eta_l^{\nu,(m+1)}\eta_l^{2,(m+1)}. \end{aligned}$$

By using Eq. (C·7), we find that the above equation becomes $2^{m+1}\delta_{2,\nu}$. Therefore,

$$= \quad \begin{cases} 2^{m+1}, & (\mu,\nu) = (1,2), \\ 0, & \mu = 1, \nu \neq 1, 2. \end{cases}$$

This completes the proof.

Appendix D: Derivation of All Solutions of the SPEs for $p \leq 3$

D.1 Case of $p = 2$

Because of the rotational symmetry, $R_{1I} = 0$ is assumed. There are three variables, R_{1R}, R_{2R}, and R_{2I}. Without loss of generality, hereafter we assume $R_{1R} > 0$. When $p = 2$, the probability P_l is

$$\begin{aligned} P_1 \quad &= \quad P_3 = \frac{1+a}{4}, \\ P_2 \quad &= \quad P_4 = \frac{1-a}{4}. \end{aligned}$$

By definition, $c_{\mu\nu}$ and Ξ_l^2 are

$$c_{11} = 2P_1u_1 + 2P_2u_2 = c_{22}, \tag{D·1}$$

$$c_{12} = 2P_1u_1 - 2P_2u_2 = c_{21}, \tag{D·2}$$

$$\Xi_1^2 = R^2 + 2R_{1R}R_{2R}, \tag{D·3}$$

$$\Xi_2^2 = R^2 - 2R_{1R}R_{2R}. \tag{D·4}$$

The SPEs are

$$R_{1R} = \beta J(c_{11}R_{1R} + c_{12}R_{2R}), \tag{D·5}$$

$$R_{2R} = \beta J(c_{12}R_{1R} + c_{11}R_{2R}), \tag{D·6}$$

$$R_{1I} = \beta J(c_{11}R_{1I} + c_{12}R_{2I}), \tag{D·7}$$

$$R_{2I} = \beta J(c_{12}R_{1I} + c_{11}R_{2I}). \tag{D·8}$$

I. $R_2 = 0$. Memory pattern: M

From the above equations, $\Xi_1 = \Xi_2 = R$, $\beta J c_{11} = 1$, and $c_{12} = 0$ follow. From these, $x_1 = x_2$ and $u_1 = u_2$ follow. Thus, from $c_{12} = 0$, $P_1 = P_2$ is derived. Thus, the memory pattern exists only for $a = 0$. The critical temperature is obtained from $u_1(0) = \frac{1}{\beta J}$, that is, $T_c^{(M)} = \frac{J}{2}$. Therefore, Eqs. (37)-(39) in the main text follow.

II. $R_{2I} \neq 0$. Continuous attractor: CA

From Eq. (D·8), $c_{11} = \frac{1}{\beta J}$ follows. Substituting this into Eq. (D·6), because $R_{1R} \neq 0$, $c_{12} = 0$ follows. Using these relations, from Eqs. (D·1) and (D·2), we obtain $P_1u_1 = P_2u_2$ and

$$u_1 = \frac{1}{(1+a)\beta J}, \tag{D·9}$$

$$u_2 = \frac{1}{(1-a)\beta J}. \tag{D·10}$$

From Eqs. (D·3) and (D·4), we obtain

$$\Xi_1^2 + \Xi_2^2 = 2R^2.$$

Therefore,

$$R = \frac{\sqrt{\Xi_1^2 + \Xi_2^2}}{\sqrt{2}} = \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{2}\beta J}. \tag{D·11}$$

If $a = 0$, $u_1 = u_2$ and $x_1 = x_2$ follow. Thus, $R = \frac{x_1}{\beta J} = \Xi_1$. From Eq. (D·3), we obtain

$R_{1R}R_{2R} = 0$. Since $R_{1R} > 0$, $R_{2R} = 0$ follows. Therefore,

$$R^2 = R_{1R}^2 + R_{2I}^2.$$

Thus, either R_{1R} or R_{2I} can freely change, that is, this solution is a one-parameter family. Therefore, it is a continuous solution. Next, we consider the case of $a \neq 0$. From Eqs. (D·3) and (D·4), R_{2R} is expressed as

$$R_{2R} = \frac{1}{4R_{1R}}(\Xi_1^2 - \Xi_2^2).$$

From the definition of R, R_{2I} is

$$R_{2I}^2 = R^2 - R_{1R}^2 - R_{2R}^2.$$

Thus, R_{2R} and R_{2I} are functions of R_{1R}. From the condition $R_{2I}^2 \geq 0$, we obtain

$$\frac{\Xi_1 - \Xi_2}{2} \leq R_{1R} \leq \frac{\Xi_1 + \Xi_2}{2}. \tag{D·12}$$

Since this solution is a one-parameter family, it is a continuous solution. The critical temperature is determined by $u_2(0) = \frac{1}{(1-a)\beta J}$. That is, $T_c^{(\mathrm{CA})} = \frac{(1-a)J}{2}$.

III. $R_2 \neq 0$, $R_{2I} = 0$

Because R_{1R} and $R_{2R} \neq 0$, from Eqs. (D·5) and (D·6), we obtain

$$\{\beta J(c_{11} + c_{12}) - 1\}\{\beta J(c_{11} - c_{12}) - 1\} \quad = \quad 0.$$

We study the two cases of A $\beta J(c_{11} + c_{12}) = 1$ and B $\beta J(c_{11} - c_{12}) = 1$ separately.

III-A. Case of $\beta J(c_{11} + c_{12}) = 1$

By adding Eqs. (D·1) and (D·2), we obtain $c_{11} + c_{12} = 4P_1 u_1$. Thus, we have

$$u_1 \quad = \quad \frac{1}{(1+a)\beta J}. \tag{D·13}$$

From this, x_1 is determined. By using $\beta J(c_{11} + c_{12}) = 1$, Eq. (D·5) becomes

$$c_{12}(R_{1R} - R_{2R}) = 0.$$

We study the two cases of A-1 $c_{12} = 0$ and A-2 $R_{1R} = R_{2R}$ separately.

III-A-1. $c_{12} = 0$. Continuous attractor: CA

From Eq. (D·5), we obtain $c_{11} = \frac{1}{\beta J}$. Therefore, we have two conditions, $c_{11} = \frac{1}{\beta J}$ and $c_{12} = 0$, as in case II. Thus, this is the continuous solution and Eqs. (D·9) and (D·10) hold. In this case, we have

$$R^2 = R_{1R}^2 + R_{2R}^2. \tag{D·14}$$

III-A-2. $R_{1R} = R_{2R}$. Symmetric mixed solution: S_1

Since $R_{1I} = R_{2I} = 0$, we obtain

$$R_1 = R_2 = \frac{x_1}{2\beta J},\ R = \frac{x_1}{\sqrt{2}\beta J}. \tag{D·15}$$

From these relations, $x_2 = 0$, $u_2 = \frac{1}{2}$, and Eq. (43) follow. From Eq. (D·13), the critical temperature is $T_{\rm c}^{({\rm S}_1)} = \frac{(1+a)J}{2}$.

III-B. $\beta J(c_{11} - c_{12}) = 1$.

From Eqs. (D·1) and (D·2), we obtain $c_{11} - c_{12} = 4P_2u_2$. Thus, we have

$$u_2 = \frac{1}{(1-a)\beta J}. \tag{D·16}$$

By using $\beta J(c_{11} - c_{12}) = 1$, Eq. (D·5) becomes

$$c_{12}(R_{1R} + R_{2R}) = 0.$$

We study the two cases of B-1 $c_{12} = 0$ and B-2 $R_{1R} = -R_{2R}$ separately.

III-B-1. $c_{12} = 0$**. Continuous attractor**

Since $c_{11} = \frac{1}{\beta J}$ follows, this is the CA.

III-B-2. $R_{1R} = -R_{2R}$**. Symmetric mixed solution: S**$_2$

Since $R_{1R} = -R_{2R}$ and $R_{1I} = R_{2I} = 0$, we obtain

$$R^2 = R_1^2 + R_2^2 = 2R_1^2. \tag{D·17}$$

From Eqs. (D·3) and (D·4), we obtain

$$\Xi_1^2 = R^2 - 2R_1^2 = 0, \tag{D·18}$$

$$\Xi_2^2 = R^2 + 2R_1^2 = 2R^2. \tag{D·19}$$

Thus, $x_1 = 0$ because $\Xi_l = \frac{x_l}{\beta J}$. Thus, $u_1 = 1/2$. Therefore,

$$R = \frac{x_2}{\sqrt{2}\beta J}, \tag{D·20}$$

$$R_1 = R_2 = \frac{x_2}{2\beta J}. \tag{D·21}$$

The critical point is $T_{\rm c}^{({\rm S}_2)} = \frac{(1-a)J}{2}$.

D.2 Case of $p = 3$

Because of the rotational symmetry, $R_{1I} = 0$ is assumed. There are five variables, $R_{1R}, R_{2R}, R_{2I}, R_{3R}$, and R_{3I}. Hereafter, we assume $R_{1R} > 0$ without loss of generality.

When $p = 3$, the probability P_l is

$$P_1 = \frac{1+3a}{8} = P_5,$$
$$P_2 = \frac{1-a}{8} = P_3 = P_4 = P_6 = P_7 = P_8.$$

From the definition of $c_{\mu\nu}$ and Ξ_l, we obtain

$$c_{11} = 2P_1u_1 + 2P_2(u_2 + u_3 + u_4) = c_{22} = c_{33}, \quad \text{(D·22)}$$
$$c_{12} = 2P_1u_1 + 2P_2(u_2 - u_3 - u_4) = c_{21}, \quad \text{(D·23)}$$
$$c_{13} = 2P_1u_1 + 2P_2(-u_2 - u_3 + u_4) = c_{31}, \quad \text{(D·24)}$$
$$c_{23} = 2P_1u_1 + 2P_2(-u_2 + u_3 - u_4) = c_{32}, \quad \text{(D·25)}$$
$$\Xi_1^2 = R^2 + 2a' + 2b' + 2c', \quad \text{(D·26)}$$
$$\Xi_2^2 = R^2 + 2a' - 2b' - 2c', \quad \text{(D·27)}$$
$$\Xi_3^2 = R^2 - 2a' - 2b' + 2c', \quad \text{(D·28)}$$
$$\Xi_4^2 = R^2 - 2a' + 2b' - 2c', \quad \text{(D·29)}$$

where

$$a' = R_{1R}R_{2R} + R_{1I}R_{2I}, \quad \text{(D·30)}$$
$$b' = R_{1R}R_{3R} + R_{1I}R_{3I}, \quad \text{(D·31)}$$
$$c' = R_{2R}R_{3R} + R_{2I}R_{3I}. \quad \text{(D·32)}$$

The SPEs become

$$R_{1R} = \beta J(c_{11}R_{1R} + c_{12}R_{2R} + c_{13}R_{3R}), \quad \text{(D·33)}$$
$$R_{2R} = \beta J(c_{12}R_{1R} + c_{11}R_{2R} + c_{23}R_{3R}), \quad \text{(D·34)}$$
$$R_{3R} = \beta J(c_{13}R_{1R} + c_{23}R_{2R} + c_{11}R_{3R}), \quad \text{(D·35)}$$
$$R_{1I} = 0, \quad \text{(D·36)}$$
$$R_{2I} = \beta J(c_{11}R_{2I} + c_{23}R_{3I}), \quad \text{(D·37)}$$
$$R_{3I} = \beta J(c_{23}R_{2I} + c_{11}R_{3I}). \quad \text{(D·38)}$$

I. $(R_2, R_3) = (0, 0)$**. Memory pattern: M**

From the SPEs, $c_{11} = \frac{1}{\beta J}$ and $c_{12} = c_{13} = 0$ follow. Since $a' = b' = c' = 0$, $\Xi_1 = \Xi_2 = \Xi_3 = \Xi_4 = R$, $u_1 = u_2 = u_3 = u_4$, and $c_{23} = 0$ follow. Thus, the memory pattern exists only for $a = 0$ and Eqs. (46) and (47) are derived. The critical temperature is $T_{\rm c}^{\rm (M)} = \frac{J}{2}$.

II. $(R_{2I}, R_{3I}) \neq (0, 0)$

From Eqs. (D·37) and (D·38), we obtain

$$(1 - \beta J c_{11})^2 - (-\beta J c_{23})^2 = 0, \tag{D·39}$$

$$\{\beta J(c_{11} + c_{23}) - 1\}\{\beta J(c_{11} - c_{23}) - 1\} = 0. \tag{D·40}$$

Since $R_{1R} > 0$, from Eqs. (D·33)-(D·35) and Eq. (D·39), we obtain

$$-(\beta J c_{11} - 1)(c_{12}^2 + c_{13}^2) + 2\beta J c_{12} c_{13} c_{23} = 0. \tag{D·41}$$

We study the two cases of A $\beta J(c_{11} + c_{23}) = 1$ and B $\beta J(c_{11} - c_{23}) = 1$ separately.

II-A. $\beta J(c_{11} + c_{23}) = 1$**.**

By using $\beta J(c_{11} + c_{23}) = 1$, Eq. (D·41) becomes

$$c_{23}(c_{12} + c_{13})^2 = 0. \tag{D·42}$$

We study the two cases of A-1 $c_{23} = 0$ and A-2 $c_{23} \neq 0$ separately.

II-A-1. $c_{23} = 0$**. Continuous attractor: CA**

From Eqs. (D·37) and (D·38), we obtain $c_{11} = \frac{1}{\beta J}$. From Eqs. (D·34) and (D·35), we obtain $c_{12} = c_{13} = 0$. From $c_{12} = c_{13} = c_{23} = 0$, we obtain $u_2 = u_3 = u_4$ and $P_1 u_1 = P_2 u_2$. From Eq. (D·22), we obtain $8P_1 u_1 = c_{11}$. Since $c_{11} = \frac{1}{\beta J}$, we obtain

$$u_1 = \frac{1}{8P_1\beta J} = \frac{1}{(1+3a)\beta J}, \tag{D·43}$$

$$u_2 = \frac{P_1}{P_2} u_1 = \frac{1}{(1-a)\beta J}. \tag{D·44}$$

From Eqs. (D·43) and (D·44), x_1 and $x_2 = x_3 = x_4$ are uniquely determined. From the relation $x_l = \beta J \Xi_l$, Ξ_l is determined. From Eqs. (D·27)-(D·29), we obtain $a' = b' = c'$. Thus, we have

$$\Xi_1^2 = R^2 + 6a', \tag{D·45}$$

$$\Xi_2^2 = R^2 - 2a' = \Xi_3^2 = \Xi_4^2. \tag{D·46}$$

Subtracting both sides of Eq. (D·46) from those of Eq. (D·45), we obtain

$$a' = \frac{x_1^2 - x_2^2}{8(\beta J)^2}. \tag{D·47}$$

Because $a' = b'$, we have

$$R_{2R} = \frac{a'}{R_{1R}} = \frac{b'}{R_{1R}} = R_{3R}. \tag{D·48}$$

On the other hand, adding both sides of Eq. (D·45) to those of Eq. (D·46), we obtain

$$R^2 = \frac{1}{2}(\Xi_1^2 + \Xi_2^2 - 4a') = \frac{x_1^2 + 3x_2^2}{4(\beta J)^2}. \tag{D·49}$$

From Eq. (D·48) and $a' = b' = c' = R_{2R}^2 + R_{2I}R_{3I}$, we obtain

$$R_{2I}R_{3I} = a' - R_{2R}^2.$$

In addition, from the definition of R^2, we obtain $R_{2I}^2 + R_{3I}^2 = R^2 - R_{1R}^2 - 2R_{2R}^2$. Thus, we obtain

$$R_{2I}^4 + (R_{1R}^2 + 2R_{2R}^2 - R^2)R_{2I}^2 + (a' - R_{2R}^2)^2 = 0.$$

Since R_{2I}^2 and R_{3I}^2 satisfy the same equation, assuming $R_{2I}^2 \geq R_{3I}^2$, we obtain

$$R_{2I}^2 = \frac{-(R_{1R}^2 + 2R_{2R}^2 - R^2) + \sqrt{(R_{1R}^2 + 2R_{2R}^2 - R^2)^2 - 4(a' - R_{2R}^2)^2}}{2}, \tag{D·50}$$

$$R_{3I}^2 = \frac{-(R_{1R}^2 + 2R_{2R}^2 - R^2) - \sqrt{(R_{1R}^2 + 2R_{2R}^2 - R^2)^2 - 4(a' - R_{2R}^2)^2}}{2}. \tag{D·51}$$

Thus, $R_{2R} = R_{3R}$, R_{2I}, and R_{3I} are determined by R_{1R}. Since this solution is a one-parameter family, it is a continuous solution. See Appendix F for the range of R_{1R} that is derived from the condition that R_{2I}^2 is real. Furthermore, when the correlation a is zero, we obtain $u_1 = u_2 = u_3 = u_4$ and $x_1 = x_2 = x_3 = x_4$. From Eq. (D·47), we obtain $a' = b' = c' = 0$. Thus, we obtain $R_{2R} = R_{3R} = 0$ by Eq. (D·48). In this case, we obtain $R_{2I} = 0$ or $R_{3I} = 0$ since $c' = R_{2I}R_{3I}$ becomes zero. Therefore, the number of nonzero variables among R_μ is only two.

II-A-2. $c_{23} \neq 0$**.**

From Eq. (D·42), we obtain $c_{12} + c_{13} = 0$. By Eqs. (D·23) and (D·24), we obtain $P_1u_1 = P_2u_3$. From Eqs. (D·22) and (D·25),

$$c_{11} = 4P_1u_1 + 2P_2(u_2 + u_4), \tag{D·52}$$

$$c_{23} = 4P_1u_1 + 2P_2(-u_2 - u_4). \tag{D·53}$$

Since $\beta J(c_{11} + c_{23}) = 1$, we obtain $8P_1u_1\beta J = 1$. Thus, we have

$$u_1 = \frac{1}{8\beta JP_1} = \frac{1}{(1+3a)\beta J}, \tag{D·54}$$

$$u_3 = \frac{P_1}{P_2}u_1 = \frac{1}{(1-a)\beta J}. \tag{D·55}$$

From these equations, x_1 and x_3 are uniquely determined. From Eq. (D·37), we have

$$(1-\beta Jc_{11})R_{2I}=\beta Jc_{23}R_{3I}.$$

Since $\beta J(c_{11}+c_{23})=1$, we obtain

$$R_{2I}=R_{3I}\neq 0. \tag{D·56}$$

From Eq. (D·37), we obtain $c_{12}+c_{13}=0$. Thus, from Eq. (D·33), we obtain

$$R_{1R}=\frac{c_{12}}{c_{23}}(R_{2R}-R_{3R}). \tag{D·57}$$

From Eq. (D·34), we have

$$(1-\beta Jc_{11})R_{2R}=\beta J(c_{12}R_{1R}+c_{23}R_{3R}).$$

By substituting Eq. (D·57) into Eq. (D·51), we obtain

$$(c_{23}^2-c_{12}^2)(R_{3R}-R_{2R})=0.$$

If we assume $(c_{23}^2-c_{12}^2)\neq 0$, we obtain $R_{2R}=R_{3R}$ but R_{1R} becomes zero from Eq. (D·57). Thus, we have

$$c_{23}^2-c_{12}^2=0.$$

We study the two cases of A-2-1 $c_{12}=c_{23}$ and A-2-2 $c_{12}=-c_{23}$ separately.

II-A-2-1. $c_{12}=c_{23}$**. Asymmetric mixed solution: A**$_1$

From Eqs. (D·56) and (D·57), we obtain

$$R_{2I} = R_{3I}. \tag{D·58}$$

$$R_{1R} = R_{2R}-R_{3R}. \tag{D·59}$$

From Eqs. (D·23) and (D·25), we obtain $u_2=u_3, x_2=x_3$, and $\Xi_2=\Xi_3$. From Eqs. (D·27) and (D·28), $a'=c'$ follows and we obtain

$$R_{1R}R_{2R} = R_{2R}R_{3R}+R_{2I}R_{3I}. \tag{D·60}$$

From Eqs. (D·26)-(D·29),

$$\Xi_1^2 = R^2+4a'+2b', \tag{D·61}$$

$$\Xi_2^2 = R^2-2b', \tag{D·62}$$

$$\Xi_4^2 = R^2-4a'+2b'. \tag{D·63}$$

By definition, we have

$$R^2=R_{1R}^2+R_{2R}^2+R_{2I}^2+R_{3R}^2+R_{3I}^2. \tag{D·64}$$

As is shown below, from Eqs. (78), (D·59), (D·60), (D·61), (D·62), and (D·63), the five variables are determined. Thus, this is not the CA. Equation (D·64) is expressed as

$$R^2 = R_{1R}^2 + R_{2R}^2 + 2R_{2I}^2 + (R_{2R} - R_{1R})^2$$
$$= 2R_{1R}^2 + 2R_{2R}^2 + 2R_{2I}^2 - 2a'. \quad \text{(D·65)}$$

From $b' = R_{1R}R_{3R}$ and Eq. (D·59), we obtain

$$R_{1R}^2 = a' - b'.$$

Thus, we obtain $a' > b'$. From $R_{1R}^2 = a' - b'$ and a', we obtain

$$R_{2R}^2 = \frac{a'^2}{R_{1R}^2} = \frac{a'^2}{a' - b'}.$$

By substituting R_{3R} into $a' = c'$, we obtain

$$R_{2I}^2 = a' - \frac{a'b'}{a' - b'}.$$

By substituting the above equations into Eq. (D·65), we obtain

$$R^2 = 2(2a' - b'). \quad \text{(D·66)}$$

Then, from Eq. (D·61) we have

$$\Xi_1^2 = R^2 + 4a' + 2b' = 8a', \quad \text{(D·67)}$$
$$a' = \frac{1}{8}\Xi_1^2 > 0. \quad \text{(D·68)}$$

Similarly, from Eqs. (D·62) and (D·63), we obtain

$$\Xi_2^2 = R^2 - 2b' = 4(a' - b'), \quad \text{(D·69)}$$
$$\Xi_4^2 = R^2 - 4a' + 2b' = 0. \quad \text{(D·70)}$$

Thus, $x_4 = 0$. From Eqs. (D·67) and (D·69),

$$b' = \frac{1}{8}(\Xi_1^2 - 2\Xi_2^2).$$

From Eq. (D·66),

$$R^2 = \frac{1}{4}(\Xi_1^2 + 2\Xi_2^2). \quad \text{(D·71)}$$

Since we derived $a' = c'$ and b', we obtain

$$R_{1R}^2 = a' - b' = \frac{1}{4}\Xi_2^2, \quad \text{(D·72)}$$
$$R_{2R}^2 = \frac{a'^2}{a' - b'} = \frac{\Xi_1^4}{16\Xi_2^2}, \quad \text{(D·73)}$$

$$R_{3R}^2 = (\frac{b'}{a'})^2 R_{2R}^2 = \frac{1}{16\Xi_2^2}(\Xi_1^2 - 2\Xi_2^2)^2, \tag{D·74}$$

$$\begin{aligned} R_{2I}^2 &= a' - \frac{a'b'}{a'-b'} = \frac{\Xi_1^2}{16\Xi_2^2}(4\Xi_2^2 - \Xi_1^2) \qquad \text{(D·75)} \\ &= R_{3I}^2. \end{aligned}$$

From $a' = R_{1R}R_{2R}$ and $a' = c' = \frac{1}{8}\Xi_1^2 > 0$, we obtain $R_{2R} > 0$. From $R_{2I}^2 \geq 0$, we obtain the following condition:

$$\begin{aligned} (2\Xi_2 + \Xi_1)(2\Xi_2 - \Xi_1) &\geq 0, \\ \Xi_1 &\leq 2\Xi_2. \end{aligned}$$

From the definition of c' and $c' = \frac{1}{8}\Xi_1^2$, we obtain

$$\begin{aligned} R_{2R}R_{3R} &= \frac{1}{8}\Xi_1^2 - R_{2I}^2 \\ &= \frac{\Xi_1^2}{16\Xi_2^2}(\Xi_1 + \sqrt{2}\Xi_2)(\Xi_1 - \sqrt{2}\Xi_2). \end{aligned}$$

From the condition $\Xi_1 \leq 2\Xi_2$, we obtain $R_{2R}R_{3R} \leq 0$. Thus, $R_{3R} \leq 0$. The critical point is $T_c^{(A_1)} = \frac{(1-a)J}{2}$. The values of u_l , R_{lR}, R_{lI}, and R are

$$\begin{aligned} u_1 &= \frac{1}{(1+3a)\beta J}\ , \quad u_2 = u_3 = \frac{1}{(1-a)\beta J}\ , \quad u_4 = \frac{1}{2}\ , \ x_4 = 0, \\ R_{1R} &= \frac{1}{2}\Xi_2,\ R_{2R} = \frac{1}{4}\frac{\Xi_1^2}{\Xi_2},\ R_{3R} = -\frac{1}{4\Xi_2}|\Xi_1^2 - 2\Xi_2^2|, \\ R_{2I}^2 &= \frac{\Xi_1^2}{16\Xi_2^1}(4\Xi_2^2 - \Xi_1^2) = R_{3I}^2,\ R_{2I} = R_{3I},\ R = \frac{1}{2}\sqrt{\Xi_1^2 + 2\Xi_2^2}. \end{aligned}$$

II-A-2-2. $c_{12} = -c_{23}$**. Asymmetric mixed solution: A$_2$**

The asymmetric mixed solution A_2 is obtained from the condition $c_{23} = -c_{12}$. This solution is derived from solution A_1 by replacing $\mu = 3$ with 2, $l = 2$ with 3, and $l = 4$ with 2.

II-B. $\beta J(c_{11} - c_{23}) = 1$**.**

By using $\beta J(c_{11} - c_{23}) = 1$, Eq. (D·41) becomes

$$c_{23}(c_{12} - c_{13})^2 = 0. \tag{D·76}$$

We study the two cases of B-1 $c_{23} = 0$ and B-2 $c_{23} \neq 0$ separately.

II-B-1. $c_{23} = 0$**. Continuous attractor: CA**

From the conditions, the solution is the CA.

II-B-2. $c_{23} \neq 0$**.**

From Eq. (D·76), we obtain $c_{12} - c_{13} = 0$. From Eqs. (D·23) and (D·24), we obtain

$u_2 = u_4$. By using $\beta J(c_{11} - c_{23}) = 1$ and Eqs. (D·22) and (D·25), we obtain

$$u_2 = \frac{1}{8\beta J P_2} \quad = \quad \frac{1}{(1-a)\beta J} = u_4. \tag{D·77}$$

From this, $x_2 = x_4$ is uniquely determined. From Eq. (D·37),

$$(1 - \beta J c_{11}) R_{2I} = \beta J c_{23} R_{3I}. \tag{D·78}$$

Since $\beta J(c_{11} - c_{23}) = 1$, we obtain $R_{2I} = -R_{3I} \neq 0$. From Eq. (D·33), we obtain

$$R_{1R} = -\frac{c_{12}}{c_{23}}(R_{2R} + R_{3R}). \tag{D·79}$$

From Eq. (D·34),

$$(1 - \beta J c_{11}) R_{2R} = \beta J (c_{12} R_{1R} + c_{23} R_{3R}). \tag{D·80}$$

By substituting Eq. (D·79) into Eq. (D·80), we obtain

$$(c_{23}^2 - c_{12}^2)(R_{2R} + R_{3R}) = 0.$$

If we assume $(c_{23}^2 - c_{12}^2) \neq 0$, we obtain $R_{2R} = -R_{3R}$ but R_{1R} becomes zero because of Eq. (D·79). Thus, we have

$$c_{23}^2 - c_{12}^2 = 0.$$

We study the two cases of B-2-1 $c_{12} = c_{23}$ and B-2-2 $c_{12} = -c_{23}$ separately.

II-B-2-1. $c_{12} = -c_{23}$**. Asymmetric mixed solution: A**$_3$

Similarly to the case of II-A-2-1, we obtain

$$u_1 = \frac{1}{(1+3a)\beta J},\ u_2 = u_4 = \frac{1}{(1-a)\beta J},\ u_3 = \frac{1}{2},\ x_3 = 0, R = \frac{\sqrt{\Xi_1^2 + 2\Xi_2^2}}{2},$$
$$R_{1R} = \frac{1}{2}\Xi_1,\ R_{2R} = \frac{\Xi_1}{4} = R_{3R},\ R_{2I}^2 = \frac{1}{16}(4\Xi_2^2 - \Xi_1^2) = R_{3I}^2,\ R_{2I} = -R_{3I}.$$

Thus, $2\Xi_2 \geq \Xi_1$ should hold. The critical point is $T_{\rm c}^{({\rm A}_3)} = \frac{(1-a)J}{2}$.

II-B-2-2. $c_{12} = c_{23}$**. Symmetric mixed solution: S**$_3$

Similarly to the case of II-A-2-1, we obtain the following:

$$R_{2R} = R_{3R}\ ,\ R_{2I} = -R_{3I}.$$

The critical temperature is $T_{\rm c}^{({\rm S}_3)} = \frac{(1-a)J}{2}$. Since $R_{1R} > 0$, R_{1R} becomes $\frac{1}{2}\Xi_2$. Since $a' = R_{1R}R_{2R}$, we obtain $a' = b' = c' < 0$. Since $R_{2R} = R_{3R} < 0$, we obtain $R_{2R} = -\frac{\Xi_2}{4}$. Thus, we obtain

$$u_1 = \frac{1}{2},\ x_1 = 0,\ u_2 = u_3 = u_4 = \frac{1}{(1-a)\beta J},$$

$$R=\frac{\sqrt{3}}{2}\Xi_2,\ R_{1R}=\frac{1}{2}\Xi_2,\ R_{2R}=-\frac{\Xi_2}{4}=R_{3R},\ R_{2I}^2=\frac{3}{16}\Xi_2^2=R_{3I}^2,\ R_{2I}=-R_{3I}.$$

$R_1=R_2=R_3$ holds. The critical point is $T_c^{(S_3)}=\frac{(1-a)J}{2}$.

III. $(R_{2I},R_{3I})=(0,0)$ **Symmetric mixed solution:** $\mathbf{S}_4$

Firstly, we assume $R_{1R}=R_{2R}=R_{3R}>0$. Thus, $R_1=R_{1R}=R_2=R_3$ follows. From Eqs. (D·30)-(D·32), $a'=b'=c'=R_1^2$ follows. We assume $R_{1R}=R_{2R}=R_{3R}$. We obtain

$$R^2=R_1^2+R_2^2+R_3^2=3R_1^2. \tag{D·81}$$

From Eq. (D·26), we obtain $x_1=3\beta JR_1$. Thus, we have

$$R_1 = \frac{x_1}{3\beta J}=R_2=R_3, \tag{D·82}$$

$$R = \frac{x_1}{\sqrt{3}\beta J}. \tag{D·83}$$

From Eqs. (D·27)-(D·29), we obtain

$$x_2=x_3=x_4=\beta JR_1\ <\ x_1=3\beta JR_1.$$

Adding both sides of the SPEs. (D·33)-(D·35) and using Eqs. (D·22)-(D·25), we obtain

$$3 = \beta J(18P_1u_1+2P_2u_2+2P_2u_3+2P_2u_4).$$

Because $u_2=u_3=u_4$, we obtain

$$\frac{1}{\beta J} = 6P_1u_1+2P_2u_2.$$

From the relations $x_1=3x_2=3x_3=3x_4$, $u_l=u(x_l)$, and $R=\frac{x_1}{\sqrt{3}\beta J}$, the identity $R^2=\frac{2}{\beta J}\sum_{l=1}^{2^{p-1}}P_lu_lx_l^2$ becomes

$$\frac{1}{\beta J} = \frac{3}{4}(1+3a)u(x_1)+\frac{1}{4}(1-a)u(\frac{x_1}{3}). \tag{D·84}$$

Therefore, x_1 is determined by Eq. (D·84). Let us derive the critical point of the symmetric mixed solution S_4 from Eq. (D·84). The function $u(x)$ decreases monotonically as x increases and takes a maximum value of $\frac{1}{2}$ at $x=0$. Substituting $u(0)=\frac{1}{2}$ into Eq. (D·84), we obtain the critical point $T_c^{(S_4)}=\frac{(1+2a)J}{2}$. From the definition of $c_{\mu\nu}$, Eq. (55) is derived. Thus, Eqs. (52)-(56) are derived.

Now, we show that the case that one or two of the $R_{\mu R}$ have the opposite sign does not satisfy the SPEs. Let us consider the case that $R_{1R}=R_{2R}=-R_{3R}>0$. In this case, $a'=-b'=-c'=R_1^2$. Thus, $\Xi_1^2=\Xi_3^2=\Xi_4^2=R^2-2a'=R_1^2$ and $x_1=x_3=x_4=\beta JR_1$ follow. That is, $u_1=u_3=u_4$ holds. From Eqs. (D·33) and (D·34), we obtain

$$1 = \beta J(c_{11}+c_{12}-c_{13}), \tag{D·85}$$

$$1 = \beta J(c_{12} + c_{11} - c_{23}). \tag{D·86}$$

Thus, $c_{12} = c_{23}$ follows. Substituting the definitions of c_{12} and c_{13} into this relation, we obtain $u_3 = u_4$. From Eq. (D·35), we obtain

$$1 = \beta J(c_{11} - 2c_{13}). \tag{D·87}$$

Thus, $c_{12} = -c_{13}$ follows. From this, we obtain $u_1 = \frac{P_2}{P_1}u_3$. Since $u_1 = u_3$, this holds only for $a = 0$. Finally, let us consider the case that $R_{1R} = -R_{2R} = -R_{3R} > 0$. Similarly, we obtain $u_1 = u_2 = u_4$ and $u_1 = \frac{P_2}{P_1}u_2$. Thus, this holds only for $a = 0$.

Appendix E: Stability Analysis of Irrelevant Solutions for $p \leq 3$

Now, we investigate the Hessian matrix at each unstable solution. Each component of the Hessian matrix is given in Eqs. (61)-(63).

E.1 Case of $p = 2$

E.1.1 Symmetric mixed solution S_2

For S_2, we have

$$u_1 = \frac{1}{2}\ , \quad u_2 = \frac{1}{(1-a)\beta J}, \quad R_{1R} = \frac{x_1}{2\beta J} = -R_{2R}\ , \quad R_{2I} = 0, \quad R = \frac{x_1}{\sqrt{2}\beta J}.$$

Thus, the critical point is $T_{\rm c}^{(\mathrm{S}_2)} = \frac{(1-a)J}{2}$. The values of $c_{\mu\mu}$ and $c_{\mu\nu}$ are

$$c_{\mu\mu} = \frac{1}{2\beta J} + \frac{1+a}{4}, \quad c_{\mu\nu} = -\frac{1}{2\beta J} + \frac{1+a}{4}, \qquad (\mu \neq \nu).$$

Now, we put $\tilde{\gamma} \equiv JN(\frac{1}{2} - \frac{1+a}{4}\beta J)$. Therefore, the Hessian matrix $\mathcal{H}$ is expressed as

$$\mathcal{H} = \begin{array}{c} \\ 1R \\ 2R \\ 1I \\ 2I \end{array} \begin{array}{c} \begin{array}{cccc} 1R & 2R & 1I & 2I \end{array} \\ \begin{pmatrix} \tilde{A} & -\tilde{A} & 0 & 0 \\ -\tilde{A} & \tilde{A} & 0 & 0 \\ 0 & 0 & \tilde{\gamma} & -\tilde{\gamma} \\ 0 & 0 & -\tilde{\gamma} & \tilde{\gamma} \end{pmatrix} \end{array},$$

where $\tilde{A} = \tilde{\gamma} - 2JN(\beta J)^2 X_2 R_{1R}^2$. Its determinant is

$$|\mathcal{H} - \lambda E| = (2\tilde{A} - \lambda)(2\tilde{\gamma} - \lambda)(-\lambda)^2.$$

The eigenvalues of this matrix are

$$\lambda = 0 \text{ (2-fold)}\ ,\ 2\tilde{A},\ 2\tilde{\gamma}.$$

If $\tilde{\gamma} > 0$, the solution is stable. The condition for this is $T > \frac{(1+a)J}{2}$. However, the condition for the existence of the solution S_2 is $T < T_c^{(S_2)} = \frac{(1-a)J}{2}$. Therefore, the symmetric mixed solution S_2 is unstable.

E.2 Case of p = 3

E.2.1 Symmetric mixed solution S_3

For S_3, we have

$$u_1 = \frac{1}{2},\ x_1 = 0,\ u_2 = u_3 = u_4 = \frac{1}{(1-a)\beta J},$$

$$R = \frac{\sqrt{3}}{2}\Xi_2,\ R_{1R} = \frac{1}{2}\Xi_2,\ R_{2R} = -\frac{\Xi_2}{4} = R_{3R},\ R_{2I}^2 = \frac{3}{16}\Xi_2^2 = R_{3I}^2,\ R_{2I} = -R_{3I}.$$

$R_1 = R_2 = R_3$ holds. The critical point is $T_c^{(S_3)} = \frac{(1-a)J}{2}$. The values of $c_{\mu\mu}$ and $c_{\mu\nu}$ are

$$c_{\mu\mu} = \frac{3}{4\beta J} + \frac{1-a}{8},\quad c_{\mu\nu} = -\frac{1}{4\beta J} + \frac{1-a}{8},\qquad (\mu \neq \nu).$$

We put $\hat{\gamma} \equiv JN(\frac{1}{4} - \frac{1-a}{8}\beta J)$. Therefore, the Hessian matrix $\mathcal{H}$ is expressed as

$$\Lambda = \begin{array}{c} \\ 1R \\ 2R \\ 3R \\ 1I \\ 2I \\ 3I \end{array}\begin{array}{c} \begin{array}{cccccc} 1R & 2R & 3R & 1I & 2I & 3I \end{array} \\ \left(\begin{array}{cccccc} 3\hat{A}-2\hat{\gamma} & -2\hat{A}+3\hat{\gamma} & -2\hat{A}+3\hat{\gamma} & 0 & \hat{B} & -\hat{B} \\ -2\hat{A}+3\hat{\gamma} & 3\hat{A}-2\hat{\gamma} & \hat{A} & \hat{B} & 0 & 0 \\ -2\hat{A}+3\hat{\gamma} & \hat{A} & 3\hat{A}-2\hat{\gamma} & -\hat{B} & 0 & 0 \\ 0 & \hat{B} & -\hat{B} & 3\hat{A}-2\hat{\gamma} & 0 & 0 \\ \hat{B} & 0 & 0 & 0 & 3\hat{A}-2\hat{\gamma} & -3\hat{A}+4\hat{\gamma} \\ -\hat{B} & 0 & 0 & 0 & -3\hat{A}+4\hat{\gamma} & 3\hat{A}-2\hat{\gamma} \end{array}\right) \end{array},$$

where $\hat{A} = \hat{\gamma} - \frac{1}{8}JN(\beta J)^2 X_2 \Xi_2^2$ and $\hat{B} = -\frac{1}{2}JN(\beta J)^2\zeta_{2R}\zeta_{2I}$. Because of the rotational symmetry, $R_{1I} = 0$ can be assumed. Then, we consider the 5×5 matrix without the $1I$ components. Its determinant is

$$|\mathcal{H} - \lambda E| \;=\; -(2\hat{\gamma}-\lambda)(4\hat{A}-4\hat{\gamma}-2\lambda)\begin{vmatrix} 3\hat{A}-2\hat{\gamma}-\lambda & -2\hat{A}+3\hat{\gamma} & 2\hat{B} \\ -4\hat{A}+6\hat{\gamma} & 4\hat{A}-2\hat{\gamma}-\lambda & 0 \\ \hat{B} & 0 & -6\hat{A}+6\hat{\gamma}+\lambda \end{vmatrix}.$$

Two of the six eigenvalues are $2\hat{\gamma}$ and $2(\hat{A}-\hat{\gamma})$. In order that S_3 is stable, $\hat{\gamma} > 0$ is necessary. The condition for this is $T > \frac{(1-a)J}{2}$. However, the condition for the existence of the solution S_3 is $T < T_c^{(S_3)} = \frac{(1-a)J}{2}$. Therefore, the symmetric mixed solution S_3 is unstable.

E.2.2 Asymmetric mixed solution A_1

For A_1, we have

$$\begin{aligned}
u_1 &= \frac{1}{(1+3a)\beta J}\ , \quad u_2 = u_3 = \frac{1}{(1-a)\beta J}\ , \quad u_4 = \frac{1}{2}\ , x_4 = 0, \\
R_{1R} &= \frac{1}{2}\Xi_2,\ R_{2R} = \frac{1}{4}\frac{\Xi_1^2}{\Xi_2},\ R_{3R} = -\frac{1}{4\Xi_2}|\Xi_1^2 - 2\Xi_2^2|, \\
R_{2I}^2 &= \frac{\Xi_1^2}{16\Xi_2^1}(4\Xi_2^2 - \Xi_1^2) = R_{3I}^2,\ R_{2I} = R_{3I},\ R = \frac{1}{2}\sqrt{\Xi_1^2 + 2\Xi_2^2}.
\end{aligned}$$

$\Xi_1 \leq 2\Xi_2$ should hold. The critical point is $T_{\mathrm{c}}^{(\mathrm{A}_1)} = \frac{(1-a)J}{2}$. The values of $c_{\mu\mu}$ and $c_{\mu\nu}$ are

$$c_{\mu\mu} = \frac{3}{4\beta J} + \frac{1-a}{8},\quad c_{12} = \frac{1}{4\beta J} - \frac{1-a}{8} = c_{23} = -c_{13}.$$

We put $\gamma' \equiv JN(\frac{1}{4} - \frac{1-a}{8}\beta J)$ and then the Hessian matrix $\mathcal{H}$ is expressed as

$$\Lambda = \begin{array}{c} \\ 1R \\ 2R \\ 3R \\ 1I \\ 2I \\ 3I \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1R & 2R & 3R & 1I & 2I & 3I \end{array} \\ \begin{pmatrix}
A' & A'-2\gamma' & B' & G' & G' & G' \\
A'-2\gamma' & A' & B'-2\gamma' & G' & G' & G' \\
B' & B'-2\gamma' & A' & G' & G' & G' \\
G' & G' & G' & C' & D' & D'+2\gamma' \\
G' & G' & G' & D' & C' & C'-2\gamma' \\
G' & G' & G' & D'+2\gamma' & C'-2\gamma' & C'
\end{pmatrix} \end{array},$$

where $A' = \gamma - \frac{1}{4}JN(\beta J)^2\{X_1\Xi_2^2 + X_2\frac{\Xi_1^4}{4\Xi_2^2}\}$, $B' = \gamma - \frac{1}{4}JN(\beta J)^2\{X_1\Xi_2^2 - X_2\frac{\Xi_1^4}{4\Xi_2^2}\}$, $C' = \gamma - JN(\beta J)^2R_{2I}^2(X_1 + X_2)$, $D' = -\gamma - JN(\beta J)^2R_{2I}^2(X_1 - X_2)$, and $G' = -\frac{1}{4}JN(\beta J)^2X_1\zeta_{1R}\zeta_{1I}$. Because of the rotational symmetry, $R_{1I} = 0$ can be assumed. Then, we consider the 5×5 matrix without the $1I$ components. Its determinant is

$$|\mathcal{H} - \lambda E| = (2\gamma' - \lambda)\begin{vmatrix}
2\gamma' - \lambda & -4\gamma' + 2\lambda & \lambda & 0 \\
2A' - 2\gamma' - \lambda & 0 & -2A' + 2B' + \lambda & 2G' \\
-2A' + 2B' + 2\gamma' + \lambda & -4\gamma' & 4A' - 4B' - \lambda & 0 \\
2G' & 0 & 0 & 2C' - 2\gamma' - \lambda
\end{vmatrix}.$$

Thus, $2\gamma'$ is one of the eigenvalues. In order that the solution is stable, $\gamma' > 0$ should hold. That is, $T > \frac{(1-a)J}{2}$ is necessary. However, the condition for the existence of the solution A_1 is $T < T_{\mathrm{c}}^{(\mathrm{A}_1)} = \frac{(1-a)J}{2}$. Therefore, the asymmetric mixed solution A_1 is unstable.

E.2.3 Asymmetric mixed solution A_3

For A_3, we have

$$u_1 = \frac{1}{(1+3a)\beta J},\ u_2 = u_4 = \frac{1}{(1-a)\beta J},\ u_3 = \frac{1}{2},\ x_3 = 0, R = \frac{\sqrt{\Xi_1^2 + 2\Xi_2^2}}{2},$$
$$R_{1R} = \frac{1}{2}\Xi_1,\ R_{2R} = \frac{\Xi_1}{4} = R_{3R},\ R_{2I}^2 = \frac{1}{16}(4\Xi_2^2 - \Xi_1^2) = R_{3I}^2,\ R_{2I} = -R_{3I}.$$

$2\Xi_2 \geq \Xi_1$ should hold. The critical point is $T_{\mathrm{c}}^{(\mathrm{A}_3)} = \frac{(1-a)J}{2}$.

The values of $c_{\mu\mu}$ and $c_{\mu\nu}$ are

$$c_{\mu\mu} = \frac{3}{4\beta J} + \frac{1-a}{8},\quad c_{12} = \frac{1}{4\beta J} - \frac{1-a}{8} = c_{13} = -c_{23}.$$

Defining $\gamma^* \equiv JN(\frac{1}{4} - \frac{1-a}{8}\beta J)$, the Hessian matrix $\mathcal{H}$ is expressed as

$$\Lambda = \begin{array}{c} \\ 1R \\ 2R \\ 3R \\ 1I \\ 2I \\ 3I \end{array} \begin{array}{c} \begin{array}{cccccc} 1R & 2R & 3R & 1I & 2I & 3I \end{array} \\ \begin{pmatrix} A^* & A^* - 2\gamma^* & -A^* - \omega^* & C^* & C^* & -C^* \\ A^* - 2\gamma^* & A^* & -A^* - \omega^* + 2\gamma^* & C^* & C^* & -C^* \\ -A^* - \omega^* & -A^* - \omega^* + 2\gamma^* & A^* & -C^* & -C^* & C^* \\ C^* & C^* & -C^* & B^* & B^* - 2\gamma^* & -B^* \\ C^* & C^* & -C^* & B^* - 2\gamma^* & B^* & -B^* + 2\gamma^* \\ -C^* & -C^* & C^* & -B^* & -B^* + 2\gamma^* & B^* \end{pmatrix} \end{array},$$

where $A^* = \gamma^* - \frac{1}{4}JN(\beta J)^2\Xi_1^2(X_1 + \frac{1}{4}X_2)$, $B^* = \gamma^* - JN(\beta J)^2 X_2 R_{2I}^2$, $C^* = -\frac{1}{4}JN(\beta J)^2 X_2 \zeta_{2R}\zeta_{2I}$, and $\omega^* = \frac{1}{2}JN(\beta J)^2 X_1 \Xi_1^2$. Because of the rotational symmetry, $R_{1I} = 0$ can be assumed. Then, we consider the 5×5 matrix without the $1I$ components. Its determinant is

$$|\mathcal{H} - \lambda E| = -(2\gamma^* - \lambda)^2 \begin{vmatrix} 1 & 2A^* - 2\gamma^* - \lambda & -\omega^* - \lambda & 2C^* \\ -2 & 0 & \lambda & 0 \\ -1 & -4A^* - 2\omega^* + 2\gamma^* + \lambda & 0 & -4C^* \\ 0 & 2C^* & 0 & 2B^* - 2\gamma^* - \lambda \end{vmatrix}.$$

Thus, $2\gamma^*$ is one of the eigenvalues. In order that the solution is stable, $\gamma* > 0$ should hold. That is, $T > \frac{(1-a)J}{2}$ is necessary. However, the condition for the existence of the solution A_3 is $T < T_{\mathrm{c}}^{(\mathrm{A}_3)} = \frac{(1-a)J}{2}$. Therefore, the asymmetric mixed solution A_3 is unstable.

Appendix F: Range of R_{1R} and Relations R_{1R}, R_{2R}, and R_{3R} for the CA when $p = 3$

In the CA studied in Appendix D, there are the following relations with $R_{1R} > 0$:

$$\begin{aligned} R_{1I} &= 0\,,\ R_{2R} = R_{3R} = \frac{a'}{R_{1R}} > 0, \\ a' &= \frac{x_1^2 - x_2^2}{8(\beta J)^2}\,,\ R^2 = \frac{x_1^2 + 3x_2^2}{4(\beta J)^2}, \\ a' &= b' = c' = R_{2R}^2 + R_{2I}R_{3I}. \end{aligned}$$

From these, we obtain $(R_{2I}R_{3I})^2 = (R_{2R} - a')^2$ and $R_{2I}^2 + R_{3I}^2 = R^2 - R_{1R}^2 - 2R_{2R}^2$. Thus, $t = R_{2R}$ or R_{3R} satisfies

$$t^2 + (R_{1R}^2 + 2R_{2R}^2 - R^2)t + (a' - R_{2R}^2)^2 = 0. \tag{F·1}$$

We put $\tilde{b} = R_{1R}^2 + 2R_{2R}^2 - R^2$ and $\tilde{c} = (a' - R_{2R}^2)^2$. Then, the solutions of Eq. (F·1) are

$$t = \frac{-\tilde{b} \pm \sqrt{\tilde{b}^2 - 4\tilde{c}}}{2}.$$

Since both R_{2I}^2 and R_{3I}^2 satisfy Eq. (F·1), we assume $R_{2I}^2 \geq R_{3I}^2$ and we have

$$\begin{aligned} R_{2I}^2 &= \frac{-\tilde{b} + \sqrt{\tilde{b}^2 - 4\tilde{c}}}{2}, \\ R_{3I}^2 &= \frac{-\tilde{b} - \sqrt{\tilde{b}^2 - 4\tilde{c}}}{2}. \end{aligned}$$

We find that $\tilde{c} = (a' - R_{2R}^2)^2 \geq 0$. Since R_{2I}^2 and R_{3I}^2 are real and non-negative, $\tilde{b}^2 - 4\tilde{c} \geq 0$ and $\tilde{b} \leq 0$ should be satisfied. Firstly, we study the range of R_{1R} in which the following relation holds:

$$\tilde{b}^2 - 4\tilde{c} \;=\; R_{1R}^4 + R^4 + 4R_{1R}^2R_{2R}^2 - 4R_{2R}^2R^2 - 2R_{1R}^2R^2 - 4a'^2 + 8a'R_{2R}^2 \geq 0. \tag{F·2}$$

Now, we put $y = R_{1R}^2$. From the relation $R_{2R} = \frac{a'}{R_{1R}}$, Eq. (F·2) reduces to

$$f(y) \equiv y^3 - 2R^2y^2 + R^4y + 8a'^3 - 4a'^2R^2 \geq 0. \tag{F·3}$$

By substituting $R^2 = \frac{\Xi_1^2 + 3\Xi_2^2}{4}$ and $a' = \frac{\Xi_1^2 - \Xi_2^2}{8}$ into Eq. (F·3), we obtain

$$f(y) = (y - \Xi_2^2)\{y - \frac{1}{4}(\Xi_1 + \Xi_2)^2\}\{y - \frac{1}{4}(\Xi_1 - \Xi_2)^2\} \geq 0. \tag{F·4}$$

Therefore, the three solutions of $f(y) = 0$ are

$$y \;=\; \Xi_2^2,\ \frac{1}{4}(\Xi_1 + \Xi_2)^2,\ \frac{1}{4}(\Xi_1 - \Xi_2)^2.$$

Next, we investigate the extreme values of this expression. The derivative of $f(y)$ is

$$f'(y) \;=\; 3y^2 - Ay + \frac{1}{16}A^2 = 3(y - \frac{1}{4}A)(y - \frac{1}{12}A), \tag{F·5}$$

where $A = \Xi_1^2 + 3\Xi_2^2$. Thus, the extreme values are attained at $y = \frac{1}{4}A$ and $\frac{1}{12}A$. Note that $R^2 = \frac{1}{4}A$. At $y = \frac{A}{4}$, f takes the following value:

$$f = -\frac{1}{16}(\Xi_1 - \Xi_2)^2\Xi_2^2,$$

and at $y = \frac{A}{12}$, it takes the value

$$f = \frac{1}{432}(\Xi_1 + 3\Xi_2)^3 - \frac{1}{16}(\Xi_1 - \Xi_2)^2\Xi_2^2.$$

We investigate the magnitude relation of the three solutions. From $\Xi_1 > \Xi_2 > 0$, we obtain $\frac{\Xi_1+\Xi_2}{2} > \Xi_2$ and $\frac{\Xi_1+\Xi_2}{2} > \frac{\Xi_1-\Xi_2}{2} > 0$. Therefore, the shape of the graph of f is as shown in Fig. F·1. Thus, the range of R_{1R} where $f \geq 0$ is satisfied is the following.

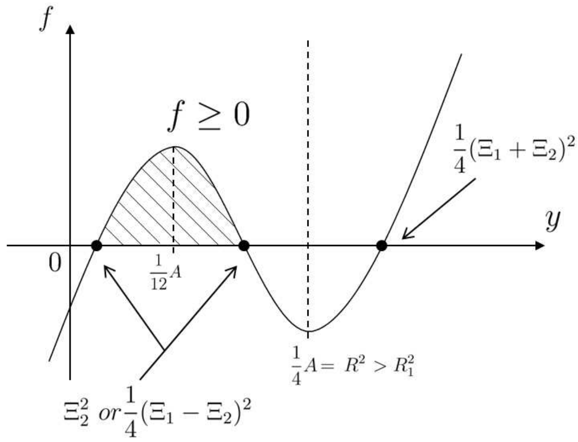

Fig. F·1. Function f.

(i) When $\Xi_1 > 3\Xi_2$, $\quad \Xi_2 \leq R_{1R} \leq \frac{1}{2}(\Xi_1 - \Xi_2)$.

(ii) When $\Xi_1 < 3\Xi_2$, $\quad \frac{1}{2}(\Xi_1 - \Xi_2) \leq R_{1R} \leq \Xi_2$.

Now, let us study the region in which $\tilde{b} \leq 0$ holds. We define the function $g(y)$ as

$$\begin{aligned} g(y) &= R_{1R}^2\tilde{b} = y^2 - R^2y + 2a'^2 \\ &= y^2 - \frac{1}{4}(\Xi_1^2 + 3\Xi_2^2)y + \frac{1}{32}(\Xi_1^2 - \Xi_2^2)^2. \end{aligned}$$

We estimate $g(y)$ at $y = \Xi_2^2$ and $\frac{1}{4}(\Xi_1 - \Xi_2)^2$.

$$\begin{aligned} g(\Xi_2^2) &= \frac{1}{32}(\Xi_1^2 - \Xi_2^2)(\Xi_1^2 - 9\Xi_2^2), \\ g(\frac{1}{4}(\Xi_1 - \Xi_2)^2) &= \frac{1}{32}(\Xi_1 - \Xi_2)^2(\Xi_1 - 3\Xi_2)(\Xi_1 + \Xi_2). \end{aligned}$$

Thus, the necessary and sufficient condition for $\tilde{b} \leq 0$ is $\Xi_1 \leq 3\Xi_2$. Therefore, case (ii) should hold. Next, let us study the magnitude relation of R_{1R}, R_{2R}, and R_{3R}. The range of R_{1R} where the CA appears is

$$R^- \leq R_{1R} \leq R^+, \tag{F·6}$$

where $R^- = \frac{\Xi_1 - \Xi_2}{2}$ and $R^+ = \Xi_2$. We compare the magnitude relation between R_{1R} and R_{2R},

$$R_{1R} - R_{2R} = \frac{1}{R_{1R}}\big(R_{1R}^2 - \frac{\Xi_1^2 - \Xi_2^2}{8}\big).$$

Since $\Xi_1 \leq 3\Xi_2$, R_{1R} takes the minimum value $\frac{1}{2}(\Xi_1 - \Xi_2)$. Thus, we obtain

$$\big(R_{1R}^2 - \frac{\Xi_1^2 - \Xi_2^2}{8}\big)_{\mathrm{min}} = \frac{1}{8}(\Xi_1 - \Xi_2)(\Xi_1 - 3\Xi_2) \leq 0.$$

Therefore, we obtain $R_{1R} \leq R_{2R}$. On the other hand, when R_{1R} takes the maximum value Ξ_2,

$$\big(R_{1R}^2 - \frac{\Xi_1^2 - \Xi_2^2}{8}\big)_{\mathrm{max}} = \frac{1}{8}(3\Xi_2 + \Xi_1)(3\Xi_2 - \Xi_1) \geq 0.$$

Thus, we obtain $R_{1R} \geq R_{2R}$. When $\Xi_1 = 3\Xi_2$, then $R_{1R} = R_{2R} = R_{3R} = \Xi_2$, $\tilde{b} = \tilde{c} = 0$, and $R_{1I} = R_{2I} = R_{3I} = 0$ follow. That is, the CA degenerates into the symmetric mixed solution S_4.

From the above results, it is proved that there is a situation with $R_{1R} = R_{2R} = R_{3R}$ as long as the CA exists, since the magnitude relation between R_{1R} and R_{2R} changes in the range of R_{1R}.