Equilibrium States and Dynamics in the Coexistence Region of Correlated Attractor and Hopfield Attractor for the Amit Model – the Case of Extensive Loading –

Tatsuya UEZU, Aya HIRANO

Graduate School of Humanities and Sciences, Nara Women's Univercity

Nara City, 630-8506, Nara, Japan

and

Masato OKADA

Brain Science Institute, Riken

Wako city, 3510198, Saitama, Japan

ABSTRACT

We study a neural network model(Amit model) which is introduced by Griniasty et.al [1] to explain the neurocognitive experiments by Miyashita et.al [2,3]. For this model, the coexistence of several attractors including correlated attractors was reported in the cases of finite and infinite loading [4,5,6]. In this paper, by means of statistical mechanical method, we study statics and dynamics of the model in the case of the extensive loading, mainly focusing on the dynamical behaviour in coexistence region of a correlated attractor and the Hopfield attractor. We derive the evolution equations by the dynamical replica theory and find several characteristic temporal behaviour. The theoretical results are confirmed by numerical simulations.

Keywords: Statistical Mechanics, Amit Model, Hopfield Attractor, Correlated Attractor, Replica Method, Dynamical Replica Theory.

1. FORMULATION

We assume that the instantaneous state of each neuron is expressed by s_i which takes ± 1, where i labels the neuron($i = 1, \cdots, N$), and time evolution is given by

$$s_i(t+1) = \mathrm{sign}(h_i(t)), \qquad (1)$$

where

$$h_i(t) = \sum_{j(\neq i)} J_{i,j} s_j(t). \qquad (2)$$

We also consider stochastic dynamics introducing temperature T, that is, the probability that $s_i(t+1)$ takes ± 1 is given by

$$\mathrm{Prob}[s_i(t+1) = \pm 1] = \frac{1 \pm \tanh(\beta h_i(t))}{2}, \qquad (3)$$

where $\beta = 1/T$.

As an extensive loading case, we consider the following synaptic weight J_{ij},

$$J_{ij} = \frac{1}{N}\{\sum_{\mu=1}^{c}(\xi_i^\mu \xi_j^\mu + a\xi_i^\mu \xi_j^{\mu-1} + a\xi_i^\mu \xi_j^{\mu+1}) + \sum_{\mu=c+1}^{p} \eta_i^\mu \eta_j^\mu\} \quad \text{for } i \neq j,$$
$$J_{ii} = 0, \quad \xi_i^0 \equiv \xi_i^c, \quad \xi_i^{c+1} \equiv \xi_i^1.$$

c is the number of condensed patterns and is set to $c = 13$. p is the total number of patterns and we consider the case that $\alpha = \frac{p}{N}$ is finite. We assume that ξ_i^μ and η_i^μ take +1 or -1 with the probability 1/2. The overlap $m_\mu(\boldsymbol{s})$ between the state of neurons $\boldsymbol{s}$ and the μ-th pattern $\boldsymbol{\xi}^\mu$ is defined as

$$m_\mu(\boldsymbol{s}) = \frac{1}{N}\sum_{i=1}^{N} s_i \xi_i^\mu \quad \text{for } \mu = 1, \cdots, c,$$
$$= \frac{1}{N}\sum_{i=1}^{N} s_i \eta_i^\mu \quad \text{for } \mu = c+1, \cdots, p.$$

The cross-talk noise $z_i(\boldsymbol{s})$ is defined as

$$z_i(\boldsymbol{s}) \equiv \sum\nolimits_{\mu>c} \eta_i^\mu m_\mu(\boldsymbol{s}).$$

As order parameters, we define $\boldsymbol{m} =^t (m_1, m_2, \cdots, m_c)$ and $r = \frac{1}{\alpha}\sum_{\mu=c+1}^{p}(m_\mu(\boldsymbol{s}))^2$ where αr is the variance of the cross-talk noise.

2. EQUILIBRIUM STATES

By using the replica method, the free energy f for the replica symmetric(RS) solution is given as [7]

$$f = \frac{1}{2}{}^t\boldsymbol{m}\boldsymbol{A}\boldsymbol{m} + \frac{\alpha\beta}{2}r(1-q) + \frac{\alpha}{2\beta}\left(\ln(1-\beta+\beta q) - \frac{\beta q}{1-\beta+\beta q}\right) - T[\int Dz \ln\{2\cosh\beta(\sqrt{\alpha r}z +^t \boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m})\}]_{\boldsymbol{\zeta}}, \tag{4}$$

where $\boldsymbol{\zeta} =^t (\zeta^1, \cdots, \zeta^c)$ and $[\Phi]_{\boldsymbol{\zeta}}$ is the average over $\boldsymbol{\zeta}$, that is $\frac{1}{2^c}\Sigma_{\{\zeta^\mu=\pm1\}}\Phi$. The S.P.E. for $\boldsymbol{m}$, q and r are given by

$$\boldsymbol{m} = [\boldsymbol{\zeta}\int Dz \tanh\beta({}^t\boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m} + \sqrt{\alpha r}z)]_{\boldsymbol{\zeta}}, \tag{5}$$

$$q = [\int Dz \tanh^2\beta({}^t\boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m} + \sqrt{\alpha r}z)]_{\boldsymbol{\zeta}}, \tag{6}$$

$$r = \frac{q}{(1-c)^2}, \tag{7}$$

where $c \equiv \beta(1-q)$ and $\boldsymbol{A}$ is a $c \times c$ matrix defined as,

$$\boldsymbol{A} \equiv \begin{pmatrix} 1 & a & 0 & \cdots & 0 & a \\ a & 1 & a & 0 & \cdots & 0 \\ 0 & a & 1 & a & 0 & 0 \\ 0 & 0 & a & 1 & a & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a & 0 & \cdots & 0 & a & 1 \end{pmatrix}. \tag{8}$$

For $T = 0$, these equations become

$$c = \sqrt{\frac{2}{\pi\alpha r}}[\exp\{-(\frac{{}^t\boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m}}{\sqrt{2\alpha r}})^2\}]_{\boldsymbol{\zeta}}, \tag{9}$$

$$\boldsymbol{m} = -2[\boldsymbol{\zeta}H(\frac{{}^t\boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m}}{\sqrt{\alpha r}})]_{\boldsymbol{\zeta}}, \tag{10}$$

$$r = \frac{1}{(1-c)^2}, \tag{11}$$

where $H(x) = \int_x^\infty \frac{dt}{\sqrt{2\pi}}e^{-t^2/2}$.

3. DYNAMICS

As for the dynamics, we derive the evolution equations for $(\boldsymbol{m}, r)$ by the dynamical replica theory(DRT) [8]. We assume that the transition probability $w_i(\boldsymbol{s})$ from a state $(s_1, \cdots, s_i, \cdots, s_N)$ to a state $F_i\boldsymbol{s} = (s_1, \cdots, -s_i, \cdots, s_N)$ takes the following form

$$w_i(\boldsymbol{s}) = \frac{1 - s_i \tanh\{\beta h_i(\boldsymbol{s})\}}{2}$$

where F_i is the flip operator of the i-th neuron. Then the master equation for the microscopic probability function $p_t(\boldsymbol{s})$ is

$$\frac{d}{dt}p_t(\boldsymbol{s}) \quad = \quad \sum_{i=1}^{N}\{w_i(F_i\boldsymbol{s})p_t(F_i\boldsymbol{s}) - w_i(\boldsymbol{s})p_t(\boldsymbol{s})\}.$$

Adopting asynchronous dynamics, we obtain the following evolution equations for $\boldsymbol{m}$ and r by using the replica method under the RS ansatz, assuming that the self-averaging for the limit of $N \to \infty$.

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{m} &= \langle \int Dx \int Dy \\
&\quad \times \boldsymbol{\zeta}\{1 - \tanh({}^t\boldsymbol{\zeta\mu} + y\sqrt{\frac{\Delta}{2\epsilon\rho}}\lambda + \frac{\lambda^2}{\sqrt{2\epsilon\rho}}x)\} \\
&\quad \times \tanh\beta({}^t\boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m} + U^-)\rangle_{\boldsymbol{\zeta}} - \boldsymbol{m}, \qquad (12) \\
\frac{1}{2}\frac{d}{dt}r &= \langle \int Dx \int Dy \\
&\quad \times \frac{1}{\alpha}\{1 - \tanh({}^t\boldsymbol{\zeta\mu} + y\sqrt{\frac{\Delta}{2\epsilon\rho}}\lambda + \frac{\lambda^2}{\sqrt{2\epsilon\rho}}x)\} \\
&\quad \times U^- \tanh\beta({}^t\boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m} + U^-)\rangle_{\boldsymbol{\zeta}} - r + 1, (13)
\end{aligned}$$

where $\epsilon = \frac{\alpha r}{2}, \Delta = \frac{\alpha\rho(1-q)}{1-\rho(1-q)}$ and $U^-(x) = \sqrt{2\epsilon}x - \Delta$. The parameters λ, ρ, q and μ are expressed in terms of $\boldsymbol{m}$ and r. The dynamical equations for $T = 0$ are obtained by replacing $\tanh\beta({}^t\boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m} + U^-)$ by $\mathrm{sign}({}^t\boldsymbol{\zeta}\boldsymbol{A}\boldsymbol{m} + U^-)$.

4. RESULTS

By solving SPE (9-11), we found the coexistence of several attractors. In fig.1(a), we show an example for which a correlated attractor and a Hopfield attractor coexist for $\alpha \lesssim 0.017$.

In a region where the correlated attractor and the Hopfield attractor coexist, we investigated dynamical behaviour. We numerically integrated the dynamical equations (12) and (13) with $T = 1/\beta = 0$ starting from the initial state $\boldsymbol{m}(0) = (m_1(0), 0, \cdots, 0)$ and $r(0) = 1$. In fig.1(b), we show the results by DRT and also the results by numerical simulations. As is seen from the figure, the basin of attraction obtained by DRT is around $m_1(0) \sim 0.5$. Except for the boundary of the basin of attraction, the results by two methods agree fairly well.

[1] M. Griniasty, M. Tsodyks and D. J. Amit,**Neural Comp.** 5(1993)1.

[2] Y. Miyashita, **Nature** 335(1988)817.

[3] Y. Miyashita and H. S. Chang, **Nature** 331(1988)68.

[4] M. Shiino, NOLTA'97(Honolulu) 665.

[5] M. Shiino and A. Yamano, NOLTA'98(Crans-Montana) 1027.

[6] M. Okada, A. Hirano and T. Uezu,

Technical Report of IEICE, NC2001-137, pp.29-36 (in Japanese).

[7] L. F. Cugliandolo and M. V. Tsodkys, **J. Phys. A: Math. Gen.** 27(1994) 741.

[8] S. N. Laughton and A. C. C. Coolen, **Phys. Rev. E** 51(1995)2581.

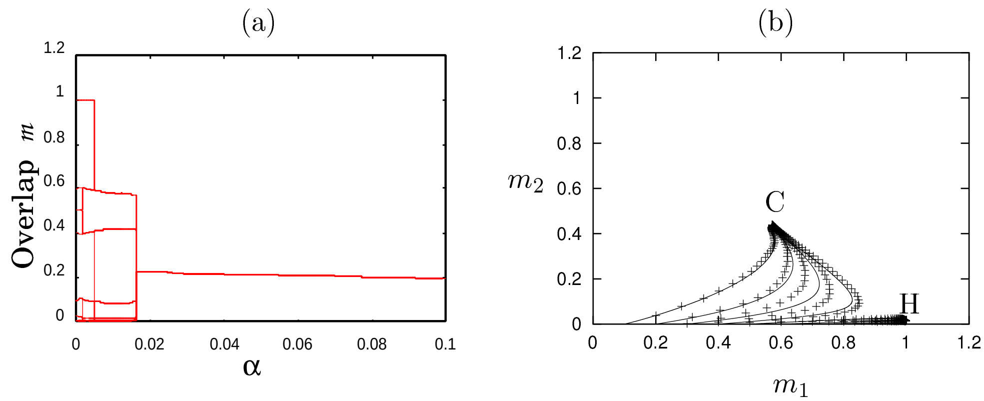

Fig. 1. $T = 0, p_c = 13$. (a) Equilibrium states. $a = 0.4$. (b) Trajectories in (m_1, m_2) plane. C: correlated attractor. H: Hopfield attractor. $a = 0.35, \alpha = 0.01$. +: simulation for $N = 30000$. Solid curve: DRT.