Correspondence between phase oscillator network and classical $XY$ model with the same random and frustrated interactions

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We study correspondence between a phase oscillator network with distributed natural frequencies and a classical $XY$ model at finite temperatures with the same random and frustrated interactions used in the Sherrington-Kirkpatrick model. We perform numerical calculations of the spin glass order parameter $q$ and the distributions of the local fields $P(R)$, where $R$ is the amplitude of the local field. As a result, we find that the parameter dependences of $P(R)$ in both models agree fairly well in all ranges of parameters in the spin glass phase and those of $q$ agree at least for lower values of parameters in the spin glass phase, if parameters are normalized by using the previously obtained correspondence relation between two models with the same other types of interactions. Furthermore, we numerically calculate the time evolution of quantities such as the instantaneous local field in the phase oscillator network in order to study the roles of synchronous and asynchronous oscillators. We also study the self-consistent equation of the local fields in the oscillator network and $XY$ model derived by the mean-field approximation.

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I. INTRODUCTION

The classical $XY$ model which describes magnetism has been studied and a lot of phase transition phenomena have been found [1]. On the other hand, there are a lot of synchronization phenomena in nature such as circadian rhythms, heartbeats, collective firing of fireflies, and so on [2,3]. For such synchronization phenomena, a phase oscillator model which describes oscillations only by phases has been proposed [4], and the synchronization-desynchronization phase transition point has been analytically obtained in the case of the uniform infinite-range interaction [5]. The models which are described only by phases are not special in the sense that the parameter dependence is trivial that the order parameters are the same, and it is not trivial whether the two models agree or not. A few years ago, for a class of infinite-range interactions with or without randomness, the order parameters exist and their SPEs and SCEs are derived for a class of infinite-range interactions with or without randomness and without frustration. On the other hand, even if interactions are infinite range, if they are random and frustrated, the correspondence between the two models has not yet been clarified. Random and frustrated interactions are very interesting in themselves, because not only is randomness ubiquitous in nature but also nontrivial phenomena such as spin glass phases exist in some models. One such interaction was studied by Sherrington and Kirkpatrick and is

\[ P(R) = \frac{1}{\sqrt{2\pi\sigma}} e^{-R^2/2\sigma^2} \]

...
called the Sherrington-Kirkpatrick (SK) model [10]. We study this interaction in this paper.

It is well known that the SK model exhibits the spin glass phase for some parameter ranges. In the spin glass phase, the total magnetization is zero, but locally each spin is frozen and has nonzero local magnetization. The spins with continuous components are also studied in Ref. [10], and the SPEs are derived and the spin glass phase is obtained. On the other hand, for the phase oscillator network, more than two decades ago, a numerical study for the SK interaction was performed by Daido and nontrivial behaviors were obtained [11]. That is, the quasientrainment state was observed, in which the substantial frequency for each oscillator is very small, but phases between two such oscillators diffuse slowly. Furthermore, the distribution of the local fields (LFs) undergoes a phase transition in which the peak position of the distribution changes from zero to nonzero value as a parameter changes and this is called the volcano transition.

In this paper, we perform numerical calculations and study the spin glass order parameter \( q \) and distributions of LFs in both models. In addition, in order to study the roles of synchronous and asynchronous oscillators in the phase oscillator network, we numerically calculate the time evolution of quantities such as phases and the local fields, and derive the SCEs of the LFs assuming that only the synchronous oscillators exist. Similarly, in the \( XY \) model, by using the naive mean-field approximation, we derive the SCEs of the LFs. We compare theoretical results with numerical ones in both models.

The structure of this paper is as follows. In Sec. II, we formulate the problem and describe the SPEs. In Sec. III, we show the results of numerical simulations. Summary and discussion are given in Sec. IV. In the Appendix, we derive the following SK interaction in common:

\[
J_{jk} = \frac{J}{\sqrt{N}} z_{jk},
\]

where \( z_{jk} \) is a random variable obeying the Gaussian distribution \( \mathcal{N}(0, 1) \). Moreover, we assume \( J_{jj} = 0 \) and \( J_{jk} = J \delta_{j,k} \).

Now, by using the replica method, we derive the SPEs for the \( XY \) model, which is originally obtained in Ref. [10].

First, in the \( XY \) model, we define the following spin glass order parameter \( q \):

\[
q = \text{Max} \left( \frac{1}{N} \sum_{j=1}^{N} e^{i(\phi_j - \phi_0)}, \frac{1}{N} \sum_{j=1}^{N} e^{i(\phi_j - (-\phi_0))} \right),
\]

where \( i = \sqrt{-1} \), \( \phi_j^\alpha (1 \leq j \leq N) \) and \( \phi_j^\beta (1 \leq j \leq N) \) are phases of two replicas \( \alpha \) and \( \beta \) that have the same interaction \( J_{jk} \). The first argument is calculated by the phase difference between \( \phi_j^\alpha \) and \( \phi_j^\alpha \), and the second argument is calculated by the phase difference between \( \phi_j^\beta \) and \( -\phi_j^\beta \). Since the Hamiltonian (1) has the reversal symmetry, that is, it is invariant under the reversal of signs of phases \( \{\phi_j\} \rightarrow \{-\phi_j\} \), we calculate the summation for the reversal phase \( -\phi_j^\beta \) shown in the second argument. Since we set \( J = 1 \), then \( q > 0 \) when the system is in the spin glass state, and \( q = 0 \) when it is in the paramagnetic state. This order parameter is exactly the same as that obtained by Sherrington and Kirkpatrick [10]. We derive the SPE for the spin glass order parameter. See the Appendix for details.

Introducing \( n \) replicas, we define the following order parameters. For \( \alpha < \beta \),

\[
q_{cc}^{\alpha\beta} = \frac{1}{N} \sum_i \cos \phi_i^\alpha \cos \phi_i^\beta, \quad q_{ss}^{\alpha\beta} = \frac{1}{N} \sum_i \sin \phi_i^\alpha \sin \phi_i^\beta,
\]

and for \( \alpha = 1, \ldots, n \)

\[
Q_{cc}^\alpha = \frac{1}{N} \sum_i \cos^2 \phi_i^\alpha, \quad Q_{ss}^\alpha = \frac{1}{N} \sum_i \sin^2 \phi_i^\alpha,
\]

\[
Q_{cc}^\alpha = \frac{1}{N} \sum_i \cos^2 \phi_i^\alpha, \quad Q_{ss}^\alpha = \frac{1}{N} \sum_i \sin^2 \phi_i^\alpha.
\]

By using the standard replica method, we obtain the disorder averaged free energy per spin \( f_{RS} = -\lim_{N \to -\infty} (\beta N)^{-1} \log Z \) by the replica method, where \( \beta = 1/k_B T \), \( k_B \) is the Boltzmann constant, and \( Z \) is the partition function. We set \( k_B = 1 \). Here, \( \cdots \) implies the average over \( \{J_{ij}\} \). Assuming the replica symmetry, we obtain

\[
f_{RS} = -\frac{1}{\beta} \frac{\beta^2 J^2}{4} (q_{cc}^2 + q_{ss}^2 + q_{cs}^2 + Q_{cc}^2 - Q_{cc}^2 - 2Q_{cs}^2) + \int Dx \int Dy \log \int d\phi M(\phi(x, y)),
\]
Thus, we obtain $Q_{cc} = Q_{as} = \frac{1}{2}$, $q_{cc} = q_{ss}$, $q_{cs} = 0$. By $\phi_j' = -\phi_j$, $\sin \phi_j \cos \phi_j$ changes its sign, and we require that $M(\phi|x, y)$ and $M(\phi'|x, -y)$ are the same functional form, and then $Q_{cs}$ is 0 follows. By the Markov chain Monte Carlo simulations, we observe that $Q_{cc} \approx Q_{as} \approx 1/2$ and $Q_{cs} \approx 0$. As for $q_{cc}$, for $\alpha < \beta$, $q_{cc} \approx q_{ss}$; $q_{cc} \approx q_{as}$, $q_{cc} \approx q_{ss}$ and $q_{cc} \approx q_{as}$ and $q_{cc} \approx q_{ss}$, or $q_{cc} \approx q_{as}$, $q_{cc} \approx q_{ss}$, or $q_{cc} \approx q_{as}$. By $\phi_j' = \pi/2 - \phi_j$, the latter becomes the former. Theoretically, both cases of $\alpha < \beta$ and $\beta < \alpha$ are taken into account for $q_{cc}$. That is, for $\alpha \neq \beta$, $q_{cc}$ should be defined as $\frac{1}{N} \sum_{j} \cos \phi_j \sin \phi_j$. Thus, numerical result $q_{cc} \approx q_{ss} \approx 0$ confirms the theoretical result $q_{cs} = 0$, and the assumption $q_{cc}q_{ss} > (\frac{1}{2}q_{ss})^2$ is satisfied. Also, if necessary, by transforming variables, the other assumptions $q_{cc} > 0$, $q_{ss} > 0$ are satisfied.

Also, numerically solving the SPEs for $q_{cc}$, $q_{ss}$, and $q_{cs}$ with $Q_{cc} = Q_{as} = \frac{1}{2}$, $Q_{ss} = 0$, we obtain $q_{cc} \approx q_{ss}$ and $q_{cs} \approx 0$. Taking these results into account, we set $q_{cc} = q_{ss}$ and $q_{cs} = q_{ss} = 0$ and obtain

$$q = 2q_{cc} \cdot$$

(14)

In the Appendix, we prove that $q$ obeys the same equation as that derived by Sherrington and Kirkpatrick [10]:

$$q = 1 - \frac{k_B T}{J} \int_{-\pi}^{\pi} dq \int_{0}^{\infty} dr e^{-\frac{r}{k_B T}} \frac{I_0\left(\frac{J}{k_B T}\sqrt{q^2 + r^2}\right)}{I_0\left(\frac{J}{k_B T}\sqrt{q^2 + r^2}\right)}. \quad (15)$$

$I_0$ is the nth order modified Bessel function of the first kind. The critical temperature is $T_c = J/2$ below which the spin glass phase appears.

### III. NUMERICAL SIMULATION

Here, we show numerical results. In this paper, we set $J= \lambda = 1$ and then $T_c = 0.5$.

#### A. Spin glass order parameter $q$

##### 1. XY model

Now, let us explain our method of numerical calculations. We use the replica exchange Monte Carlo (REM) method. We prepare 48 sets and 96 sets of temperature for $N = 100$ and 500, respectively, and a replica is assigned to each temperature. We call it a temperature replica. The temperature $T$ ranges from 0.02 to 0.96 with the increment $\Delta T = 0.02$ for $N = 100$ and $\Delta T = 0.01$ for $N = 500$, respectively. In order to calculate $q$, we prepare another set of replicas. Two sets of replicas are denoted by $\alpha$ and $\beta$, respectively. The initial values of $\phi_j$ of all replicas are set to values in $[0, 2\pi)$ randomly. For $N = 100$ (500), we exchange temperature replicas every 5000 (1000) Monte Carlo (MC) sweeps. One MC sweep corresponds to $N$ updates of spins. The number of exchanges is $10^4$. After 500 exchanges, at each temperature, we calculate the time average of $q$ using 100 sets of phases of XY spins for the last 100 MC sweeps during 5000 and 1000 MC sweeps for $N = 100$ and 500, respectively. We denote this average by $\bar{q}$. Then we take the average of $\bar{q}$ over $9500$ exchanges, which we regard as the thermal average $\langle q \rangle$. At each temperature, the sample average of $\langle q \rangle$ and its standard deviation are calculated. The number of samples is 30 and 5.
for \(N = 100\) and for \(500\), respectively. We show the results of the temperature dependence of \(q\) in Fig. 1(a) for \(N = 100\) and in Fig. 1(b) for \(N = 500\). The solid curves are the theoretical results at the thermodynamic limit of \(N = \infty\). The theoretical curves look straight, but they are slightly curved. The black circles are the sample average of \(q\) and the error bars are the standard deviation. The theoretical curves and the computer simulation results almost agree with each other at \(N = 100\) and at \(T < 0.4\) for \(N = 500\), respectively. Therefore, it is expected that the agreement between the theoretical curves and the simulation results becomes better as \(N\) is increased, and the critical temperature will be \(T_c = 0.5\), which is the theoretical result.

2. Phase oscillator network

We adopt the same definition of \(q\) by Eq. (4) as in the \(XY\) model. The computer simulation is carried out by the following method. In order to integrate Eq. (2) numerically, we adopt the Euler method with time increment \(\Delta t = 0.02\). Since the Hamiltonian is not defined for the phase oscillator network, it is impossible to use the REMC method. Therefore, in analogy to the simulated annealing method, the relaxation calculation is carried out while gradually lowering \(\sigma\) from \(\sqrt{\pi/2}\) to zero with the increment \(\Delta \sigma = 0.01\sqrt{\pi/2}\) for \(N = 100\), \(\Delta \sigma = 0.005\sqrt{\pi/2}\) for \(N = 200\), and \(\Delta \sigma = 0.000125\sqrt{\pi/2}\) for \(N = 500\). In this paper, we also call this the simulated annealing method.

At each \(\sigma\), we evolve the system until \(t = 800\) and calculate the time average of \(q\) using phases of oscillators from \(t = 501\) to \(800\) with time interval 1. We denote this by \(\bar{q}\). At each \(\sigma\), the sample average of \(\bar{q}\) and the standard deviation over samples are calculated. For this simulated annealing method, \(\omega_j (1 \leq j \leq N)\) is not generated for every \(\sigma\). Instead, first, \(\omega_j\) with \(\sigma = 1\) is generated according to \(N(0, 1)\). We denote it \(\omega_{j,0}\). Then, \(\omega_j\) with \(\sigma \neq 1\) is defined as \(\sigma \omega_{j,0}\). The initial values of \(\phi_j\) (\(1 \leq j \leq N\)) at the beginning of the simulated annealing method are chosen randomly from \([0, 2\pi]\). In the simulated annealing method, there may be cases that the relaxed state is captured at a local minimum for \(\sigma = 0\). In order to judge whether the relaxed state reaches the global minimum at \(\sigma = 0\), we use the fact that the phase oscillator network with \(\sigma = 0\) and the \(XY\) model with \(T = 0\) are the same model. Concretely, we use the following method. We prepare the same interaction for both models. In the oscillator network, we choose two replicas with \(q \approx 1\) at \(\sigma \sim 0\) obtained by the simulated annealing method. Then, we calculate \(q\) using \(\phi_j\) (\(1 \leq j \leq N\)) of one of two replicas of the phase oscillator network at \(\sigma \sim 0\). We denote it \(\phi_{j,0}\) (\(1 \leq j \leq N\)) of the \(XY\) model at the corresponding temperature \(T = \sqrt{\pi/2}\sigma\) obtained by the REMC method. If \(q > 0.99\), it is judged that the two replicas in the oscillator network reach the global minimum.

By this procedure, we obtain 100 (\(N = 100\)), 100 (\(N = 200\)), and 20 (\(N = 500\)) pairs of replicas which reach the global minimum at \(\sigma \sim 0\).

Using these pairs, we calculate the sample average of \(\bar{q}\) and the standard deviation. In Fig. 2, we display the \(\sigma\) dependence of the sample average of \(q\) with its standard deviation. The solid curve is obtained by the theoretical formula of \(q\) for the \(XY\) model by setting \(\sigma = T\sqrt{\pi/2}\). For \(\sigma < 0.4\) when \(N = 100\), \(\sigma < 0.2\) when \(N = 200\), and \(\sigma < 0.17\) when

![Figure 2](image-url)

**FIG. 2.** \(\sigma\) dependence of the sample average of \(q\) in the phase oscillator network. (a) \(N = 100\). (b) \(N = 200\). (c) \(N = 500\).

![Figure 3](image-url)

**FIG. 3.** \(\sigma\) dependence of the sample average of \(q_{\text{ave}}\) in the phase oscillator network. (a) \(N = 100\). (b) \(N = 200\). (c) \(N = 500\).
Since we have interest in behavior for long periods, it is meaningful to observe time averaged quantities.

Therefore, we introduce the following quantity \( q_{av} \) for two replicas \( \{\phi^\alpha_j\} \) and \( \{\phi^\beta_j\} \):

\[
q_{av} = \max \left( \frac{\left| \sum_{j=1}^N \bar{A}^\alpha_j \bar{A}^\beta_j e^{i(\phi^\alpha_j - \phi^\beta_j)} \right|}{\sum_{j=1}^N \bar{A}^\alpha_j \bar{A}^\beta_j}, \frac{\left| \sum_{j=1}^N \bar{A}^\alpha_j \bar{A}^\beta_j e^{i(\phi^\beta_j - \phi^\alpha_j)} \right|}{\sum_{j=1}^N \bar{A}^\alpha_j \bar{A}^\beta_j} \right),
\]

(16)

\[
\bar{A}^\alpha_j e^{i\phi^\alpha_j} = \frac{1}{T_{s}} \sum_{k=1}^{T_s} e^{i\phi^\alpha_j(k)}, \quad \bar{A}^\beta_j e^{i\phi^\beta_j} = \frac{1}{T_{s}} \sum_{k=1}^{T_s} e^{i\phi^\beta_j(k)},
\]

(17)

where \( T_s = 300 \) and \( t_k = 500 + k \). The normalization factor \( \sum_{j=1}^N \bar{A}^\alpha_j \bar{A}^\beta_j \) is determined so that \( q_{av} \) becomes 1 when \( \phi^\alpha_j \) and \( \phi^\beta_j \) are equal or their difference is constant for all \( j \). The numerical results are shown in Fig. 3 for \( N = 100, 200, \) and 500.

We note that the theoretical result of \( q \) for the XY model and numerical results of \( q_{av} \) in the phase oscillator network agree fairly well, and as \( N \) increases the coinciding range of the theoretical curve and the simulation results increases, and the critical parameter will be \( \sigma_c = T_c \sqrt{\pi/2} \) when \( N = \infty \).

The results of \( T \) dependences of \( q \) in the XY model and \( \sigma \) dependences of \( q \) in the phase oscillator network imply that they differ by the factor \( \sqrt{\pi/2} \) in the scale of abscissa axes as expected.

**B. Local field**

Now, let us study the local field \( p_j = x_j + i y_j \) which is defined by

\[
p_j = \sum_{k=1}^{N} J_{jk} e^{i\theta_k}.
\]

(18)

LFs move on the complex plane with time due to the thermal fluctuation in the XY model, and in the phase oscillator network they move on the complex plane with time according to the evolution equation (2).

**1. XY model**

We numerically examine the spatial distribution of LFs on the complex plane for all spins. The initial values of \( \phi_j \) \((1 \leq j \leq N)\) are set as the final equilibrium state obtained when we calculate \( q \). In Fig. 4, we display the distribution of LFs on the complex plane and the probability density \( P(r) \) of LFs, where \( r = \sqrt{x^2 + y^2} \). To draw Fig. 4, a Monte Carlo simulation is carried out for \( N = 500 \) and data are taken every one MC sweep during 10 000 MC sweeps. That is, 10 000 \( \times \) \( N \) data are used to draw Fig. 4. When \( T \) is low, \( P(r) \) is a volcanic shape with a hole in the center, i.e., \( r = 0 \), and the hole gradually closes with the increase of \( T \), and then it disappears and the peak position becomes \( r = 0 \) for \( T > 0.5(= T_c) \).

**2. Phase oscillator network**

We calculate LFs as in the XY model. The initial values of \( \phi_j \) \((1 \leq j \leq N)\) are set as the final state obtained when we calculate \( q \). In Fig. 5, we display the distribution of LFs

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**FIG. 4.** Local field of the XY model \((N = 500)\). Left panel: Spatial distribution of LFs on the complex plane. Right panel: Probability density of LFs, \( P(r) \). (a, b) \( T = 0.04 \). (c, d) \( T = 0.26 \). (e, f) \( T = 0.5 \). (g, h) \( T = 0.6 \).
and \( P(r) \). A computer simulation is carried out for \( N = 500 \) until \( t = 10,000 \), and data are taken every time interval 1 to draw Fig. 5. That is, the number of data to draw Fig. 5 is the same as in the \( XY \) model. As is seen from Fig. 5, with the increase of \( \sigma \) from zero, behavior of \( P(r) \) is the same as in the \( XY \) model and the peak position becomes \( r = 0 \) for \( \sigma > 0.5\sqrt{\pi/2} (= T_c \sqrt{\pi/2}) \).

![Figure 5](image1)

**FIG. 5.** The local field of the phase oscillator network (\( N = 500 \)). Left panel: Spatial distribution on the complex plane. Right panel: Probability density \( P(r) \). (a, b) \( \sigma = 0.04\sqrt{\pi/2} \). (c, d) \( \sigma = 0.26\sqrt{\pi/2} \). (e, f) \( \sigma = 0.5\sqrt{\pi/2} \). (g, h) \( \sigma = 0.6\sqrt{\pi/2} \).

3. Comparison of results for both models

In the LFs of the \( XY \) model, for \( N = 500 \), the \( T \) dependence of the radius at which the probability density has a peak is shown in Fig. 6(a). We call the radius the peak radius, and denote it by \( r_p \). The black circles show the peak radius, and the error bars show the radius at which the probability density decreases by 5% from the peak. The peak radius at \( T > 0.5(= T_c) \) becomes nearly zero. In the LFs of the phase oscillator network, for \( N = 500 \), the \( \sigma \) dependence of the peak radius is shown in Fig. 6(b). The circles and error bars have

![Figure 6](image2)

**FIG. 6.** Temperature dependence of the peak radius \( r_p \) in the \( XY \) model (\( N = 500 \)) and \( \sigma \) dependence of \( r_p \) in the phase oscillator network (\( N = 500 \)). (a) \( XY \) model. (b) Phase oscillator.

![Figure 7](image3)

**FIG. 7.** Time series of \( \sin \phi(t) \). \( N = 100 \). (a) \( J_{\beta} = 0 \). (b) \( J_{\beta} \neq 0, \sigma = 0.2 \). (c) \( J_{\beta} \neq 0, \sigma = 0.3 \).
First, we study the time evolution of \( \sin \phi \) where \( \phi \) is the phase of each oscillator. In Fig. 7, we show \( \sin \phi(t) \) of 20 oscillators for \( N = 100 \) during \( t = 0–150 \). In Fig. 7(a), we set \( J_{jk} = 0 \), that is, \( \phi_j = \omega_j t + \phi_j(0) \). In Figs. 7(b) and 7(c), we set \( J_{jk} \neq 0 \), and \( \sigma = 0.2 \sqrt{\pi/2} \) and \( 0.3 \sqrt{\pi/2} \), respectively. We note that oscillators are locked for a while and then are unlocked, and repeat this behavior. We find that, the larger \( \sigma \) is, the more fluctuations of phases there are, and trajectories behave chaotically. Next, we study trajectories of LFs for a long time for \( N = 200 \) and 500. See Figs. 8–11. We define the amplitude \( R_j \) and phase \( \Theta_j \) of the LFs by

\[
R_j e^{i \Theta_j} = p_j = \sum_k J_{jk} e^{i \phi_k}.
\]

In this simulation, we adopt the simulated annealing and the schedule is \( T_l = 0.7 - (l - 1) \times 0.02 \), \( l = 1–35 \). We obtain the following results. When \( \sigma \) is small, \( \sigma < \sigma_1 \), \( R_j \) and \( \Theta_j \) are constant or periodic depending on \( N \). The distribution of substantial frequencies is \( G(\omega) = \delta(\omega) \). When \( \sigma \) becomes large, \( \sigma_1 < \sigma < \sigma_2 \), \( R_j \) behaves chaotically, and \( \Theta_j \) has two phases. \( \Theta_j \) is almost constant in one phase, and it increases or decreases rapidly in the other phase. On average, \( \Theta_j \) evolves almost linearly. When \( \sigma \) is large enough, \( \sigma_2 < \sigma \), \( \Theta_j \) evolves almost linearly. It seems that \( \sigma_1 \) and \( \sigma_2 \) depend on \( N \). For \( N \) that we investigated, \( \sigma_1 \sim 0.1 \sqrt{\pi/2} \).

\( G(\omega) \) is one-humped and continuous, and it is impossible to separate synchronized oscillators from desynchronized ones. See Fig. 12.

C. Numerical results for several quantities in the phase oscillator network

In the phase oscillator network, in order to study the roles of synchronous and asynchronous oscillators for the correspondence, we numerically calculate several quantities. First, we study the time evolution of \( \sin \phi \). We set \( \sigma = 0.1 \sqrt{\pi/2} \) during \( t = 0–150 \). In Fig. 8(a), we set \( J_{jk} = 0 \), that is, \( \phi_j = \omega_j t + \phi_j(0) \). In Figs. 8(b) and 8(c), we set \( J_{jk} \neq 0 \), and \( \sigma = 0.2 \sqrt{\pi/2} \) and \( 0.3 \sqrt{\pi/2} \), respectively. We note that oscillators are locked for a while and then are unlocked, and repeat this behavior. We find that, the larger \( \sigma \) is, the more fluctuations of phases there are, and trajectories behave chaotically. Next, we study trajectories of LFs for a long time for \( N = 200 \) and 500. See Figs. 8–11. We define the amplitude \( R_j \) and phase \( \Theta_j \) of the LFs by

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\( G(\omega) \) is one-humped and continuous, and it is impossible to separate synchronized oscillators from desynchronized ones. See Fig. 12.
In the next subsection, we derive the self-consistent equations for LFs in the XY model and oscillator network by using approximations.

D. SCE for LFs

In the oscillator network, we derive the SCE for the case that all oscillators are synchronized. In the XY model, we derive the SCE by using the naive mean-field approximation.

1. Oscillator network

Using \( R_j \) and \( \Theta_j \), the evolution equation is rewritten as

\[
\frac{d}{dt} \psi_j = \omega_j - R_j \sin(\psi_j - \Theta_j). \tag{20}
\]

\( R_j \) and \( \Theta_j \) are constant because all oscillators are assumed to be synchronized. Thus, by defining \( \psi_j = \phi_j - \Theta_j \), we obtain

\[
\frac{d}{dt} \psi_j = \omega_j - R_j \sin \psi_j. \tag{21}
\]

The stable solution is \( \psi^*_j = \sin^{-1} \frac{\omega_j}{R_j} \), where \(|\psi^*_j| < \frac{\pi}{2}\). The probability density function of phases \( f(\psi; \omega_j) \) is \( \delta(\psi - \psi^*_j) \).

Thus, the average of \( e^{i\phi_j} \) is

\[
\langle e^{i\phi_j} \rangle = e^{i(\phi^*_j + \Theta_j)} = \left[ \sqrt{1 - \left( \frac{\omega_j}{R_j} \right)^2} + i \frac{\omega_j}{R_j} \right] e^{i\Theta_j}. \tag{22}
\]

Therefore, the SCE for LFs is

\[
R_j e^{i\Theta_j} = \sum_{j'=1}^{N} J_{jj'} \left[ \sqrt{1 - \left( \frac{\omega_{j'}}{R_{j'}} \right)^2} + i \frac{\omega_{j'}}{R_{j'}} \right] e^{i\Theta_{j'}}. \tag{23}
\]

We numerically solve the SCE (23) by iteration method. That is, from \( \{R_j\} \) and \( \{\Theta_j\} \) at iteration step \( n \), we evaluate the right-hand side of Eq. (23) to obtain \( \{R'_j\} \) and \( \{\Theta'_j\} \) at iteration step \( n+1 \). We define the distance between two configurations \( \{\phi_j\} \) and \( \{\phi'_j\} \) as

\[
d(\{\phi_j\}, \{\phi'_j\}) = \sum_{j=1}^{N} |\phi_j - \phi'_j|. \]

The convergence condition is \( d(\{\phi_j(n)\}, \{\phi_j(n+1)\}) < \epsilon \) for two successive configurations \( \{\phi_j(n)\} \) and \( \{\phi_j(n+1)\} \) with \( \epsilon = 0.01 \). It turns out that it is very difficult to obtain solutions for Eq. (23) if initial conditions are taken randomly. Then, as an initial condition, we use the numerical results obtained by the simulated annealing method, and find that almost all numerical results are solutions of the SCE when \( \sigma \) is small. For example, we find that when \( N = 100 \) and \( \sigma = 0.02\sqrt{\frac{N}{\pi}} \) all 19 configurations obtained by the simulated annealing converge by only one iteration and \( d(\{\phi_j(0)\}, \{\phi_j(1)\}) \sim 3 \times 10^{-5} \), that is, these configurations satisfy Eq. (23). We regard two configurations \( \{\phi_j\} \) and \( \{\phi'_j\} \) to be different when \( d(\{\phi_j\}, \{\phi'_j\}) > \epsilon \). We find only two different configurations among 19 configurations. When \( N = 100 \) and \( \sigma = 0.1\sqrt{\frac{N}{\pi}} \), we find that 16 configurations converge by only one iteration among 19 configurations, and all of them are regarded as the same. However, for larger values of \( \sigma \), we cannot find any solution. This is because \( R_j \) and \( \Theta_j \) are not constant and it seems that asynchronous solutions contribute to the LFs.

2. XY model

The Hamiltonian is

\[
H = -\sum_{j<k} J_{jk} \cos(\phi_k - \phi_j) = -\frac{1}{2} \sum_{j} R_j \cos(\phi_j - \Theta_j). \tag{24}
\]
Since the probability density function of phases is $P(\phi) = \frac{1}{2\pi k}\delta(\phi_j - \phi)$, defining $\psi_j = \phi_j - \Theta_j$ we obtain
\[
\langle e^{i\psi} \rangle = \frac{1}{2\pi i} \int_{0}^{2\pi} e^{i\phi_i} \frac{e^{i\psi + i\Theta}}{\sqrt{2\pi}} d\phi = \frac{1}{i} I_0(\beta R_j) e^{i\Theta_j} u(\beta R_j) \quad \text{for } R_j \neq 0.
\]
(25)

Here, $u(x) = \frac{I_1(x)}{x I_0(x)}$. Thus, we obtain
\[
R_j e^{i\Theta_j} = \sum_k J_{jk} e^{i\phi_k} = \beta \sum_k J_{jk} R_k e^{i\phi_k} u(\beta R_k).
\]
(26)

As an initial condition, we use the configuration obtained by the simulated annealing as in the oscillator network. The method to solve Eq. (26), the convergence condition, and the criterion of different solutions are the same as in the phase oscillator network. When $N = 100$ and $T = 0.02$, among 30 configurations, three configurations converge with $\epsilon = 0.01$. The numbers of iterations are rather large compared to the oscillator network and are 29, 51, and 62 for these three configurations, respectively. All of them are different, but it is difficult to distinguish these three from the figure of $j$ versus $R_j$. When $N = 100$ and $T = 0.1$, among 30 configurations, five configurations converge, and the number of iterations ranges from 50 to 70. Four configurations among five are different. We find that convergent values and initial conditions are rather different and this is consistent with the fact that the numbers of iterations are large. See Fig. 13. Therefore, in this case, final configurations by the simulated annealing for $T = 0.1$ are not considered as the solutions of the SCEs. The reason for this is considered to be that the naive mean-field approximation is not valid for high temperatures.

### IV. SUMMARY AND DISCUSSION

We summarize the results of this paper. We studied the random and frustrated interaction, the SK interaction, which is generated by the Gaussian distribution with mean zero and standard deviation $J/\sqrt{N}$. As for the distribution of natural frequencies $g(\omega)$, we adopted the Gaussian distribution with mean zero and standard deviation $\sigma$. In order to study whether correspondence between the two models exists or not, we performed numerical calculations of the spin glass order parameter $q$ and the distributions of the LFs in the $XY$ model and phase oscillator network. In the $XY$ model, we used the Markov chain Monte Carlo simulation, in particular, the REMC method and the simulated annealing method. In the oscillator network, we used the Euler method with time increment $\Delta t = 0.02$, and also used the simulated annealing method, that is, we integrated the evolution equation by decreasing $\sigma$ slowly. First, we summarize the results of $q$. In the $XY$ model, we confirmed that theoretical and numerical results agree fairly well and found that the coinciding region between the theoretical curve and the simulation results of $q$ increases as $N$ increases. For the phase oscillator network, we found that in the $\sigma$ dependence of the spin glass order parameter the numerical results agree with the theoretical curve $q[T(\sigma)]$ of the $XY$ model at least for lower values of $\sigma$ in the spin glass phase. Here, $T(\sigma) = \sqrt{2/\pi} \sigma$ is the relation obtained in the previous paper [8]. However, the coinciding region between the simulation results and the theoretical curve decreases as $N$ increases, contrary to our expectation. Since $\phi_j$ behaves intermittently in time, we introduced the order parameter $q_{av}$ for the time averaged phases, and found that the coinciding region between the theoretical curve of the $XY$ model and the simulation results of $q_{av}$ increases as $N$ increases.

Next, we summarize the results of LFs.

We studied the probability density $P(r)$ of LFs, where $r$ is the radius of the local field in the complex plane. As $T$ or $\sigma$ increases, the peak radius $r_p$ of $P(r)$ changes from a nonzero value to zero. This is the so-called volcano transition, and the transition points of the two models seem to correspond according to the relation $T = \sqrt{2/\pi} \sigma$. For the oscillator network, we numerically studied time evolution of $\sin \phi$ of each oscillator and found that oscillators are locked for a while and then are unlocked, and repeat this behavior. We also numerically studied time evolution of amplitudes $R$s and...
phases $\Theta$ of LFs. We found that when $\sigma$ is small they are constant or periodic depending on $N$, and the distribution of the substantial frequencies $G(\omega)$ is the delta function $\delta(\omega)$, but, when $\sigma$ is large, $R_j$ behaves chaotically, and $\Theta_j$ has two phases: in one phase $\Theta_j$ is almost constant, and in the other phase it increases or decreases rapidly. On average, $\Theta_j$ evolves almost linearly. $G(\omega)$ is one-humped and continuous.

Finally, we derived the SCE of LFs for the oscillator network in the case that all oscillators synchronize and for the $XY$ model by using the naive mean-field approximation. We found that for the oscillator network and $XY$ model, when $\sigma$ and $T$ are small, configurations obtained by simulated annealing satisfy the SCE, but when $\sigma$ and $T$ are large they do not. The reasons for the discrepancy between theoretical and numerical results for the LFs at large $T$ and $\sigma$ are considered to be as follows. In the oscillator network, the asynchronous oscillators do not contribute to the LFs for the solvable models when the $g(\omega)$ is one-humped and symmetric with respect to its center. However, the present results imply that asynchronous oscillators contribute to the LFs. Since $G(\omega)$ is continuous, it is difficult to separate synchronized oscillators from desynchronized ones. In the $XY$ model, the present results imply that the naive mean-field approximation is not valid except for very low temperatures. This is the same as in the case of Ising spins. The so called Onsager reaction field should be taken into account for the $XY$ model as in the Ising model. Therefore, in order to improve the present approximations for the two models, further elaborate studies are necessary, and these studies are beyond the scope of the present paper and are left as a future problem.

Here, we discuss the reason why $q_{av}$ agrees better with the theoretical result of the $XY$ model than the simple time average of $q$ in the phase oscillator network. $\{\phi_j\}$, local fields, and $\{e^{\phi_j(t)}\}$ behave intermittently in time. For larger $\sigma$ and larger $N$ in the spin glass phase, the intermittent behavior becomes stronger. As is seen from Fig. 2, the agreement between the simulation results and the theoretical curve becomes worse for larger values of $\sigma$ as $N$ is increased. Since the number of desynchronized oscillators increases as $\sigma$ increases as evidenced by Fig. 12, one of the causes of the disagreement might be that the contribution of the desynchronized oscillators to the overlap is not evaluated appropriately due to their intermittent behavior, and, as a result, the time average of the overlap between $\{e^{\phi_j(t)}\}$ and $\{e^{\phi_j(t)}\}$ is reduced. On the other hand, as noted from Fig. 3, by using the time averaged quantities $\{A_j^\alpha e^{i\phi_j}\}$ and $\{A_j^\beta e^{i\phi_j}\}$, the reduction of the overlap due to such intermittent behavior seems to weaken. However, in order to obtain a definite conclusion, more investigations on this subject are necessary. This is left as another future problem.

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**APPENDIX**

In this Appendix, we derive the disorder averaged free energy per spin and the SPEs under the ansatz of replica symmetry. The derivation is based on Ref. [12]:

$$\tilde{f} = -\frac{1}{\beta N} \lim_{N \to \infty} \lim_{\beta N h \to 0} \frac{1}{N} \log \int d\phi^1 \cdots d\phi^n e^{-\sum_{\alpha} H(\phi^\alpha)},$$

$$e^{-\beta \sum_{\alpha} H(\phi^\alpha)} = \exp \left[ \frac{\beta^2 J^2}{4N} \sum_{\alpha \beta} \left( \sum_i \cos \phi_i^\alpha \cos \phi_i^\beta \right)^2 + \left( \sum_i \sin \phi_i^\alpha \sin \phi_i^\beta \right)^2 + \left( \sum_i \cos \phi_i^\alpha \sin \phi_i^\beta \right)^2 \right] \exp \left( \sum_i \sin \phi_i^\alpha \cos \phi_i^\beta \right)^2 - N \right] \right],$$

where $\phi^\alpha = (\phi_1^\alpha, \ldots, \phi_N^\alpha)$. We define the following order parameters. For $\alpha < \beta$,

$$q_{cc}^{\alpha \beta} = \frac{1}{N} \sum_i \cos \phi_i^\alpha \cos \phi_i^\beta, \quad q_{ss}^{\alpha \beta} = \frac{1}{N} \sum_i \sin \phi_i^\alpha \sin \phi_i^\beta,$$

$$q_{cs}^{\alpha \beta} = \frac{1}{N} \sum_i \cos \phi_i^\alpha \sin \phi_i^\beta, \quad q_{sc}^{\alpha \beta} = \frac{1}{N} \sum_i \sin \phi_i^\alpha \cos \phi_i^\beta,$$

and for $\alpha = 1, \ldots, n$

$$Q_{cc}^\alpha = \frac{1}{N} \sum_i \cos^2 \phi_i^\alpha, \quad Q_{ss}^\alpha = \frac{1}{N} \sum_i \sin^2 \phi_i^\alpha, \quad Q_{cs}^\alpha = \frac{1}{N} \sum_i \cos \phi_i^\alpha \sin \phi_i^\alpha.$$

Then, we obtain

$$e^{-\beta \sum_{\alpha} H(\phi^\alpha)} = e^{-\frac{J^2}{4N} \sum_{\alpha \beta \alpha \beta} \left( \sum_i \cos \phi_i^\alpha \cos \phi_i^\beta \right)^2 + \left( \sum_i \sin \phi_i^\alpha \sin \phi_i^\beta \right)^2 + \left( \sum_i \cos \phi_i^\alpha \sin \phi_i^\beta \right)^2 + \left( \sum_i \sin \phi_i^\alpha \cos \phi_i^\beta \right)^2} \right] \exp \left[ \beta^2 J^2 N \sum_{\alpha} \left( Q_{cc}^\alpha \right)^2 + \left( Q_{ss}^\alpha \right)^2 + 2 \left( Q_{cs}^\alpha \right)^2 + \sum_{\alpha < \beta} \left( q_{cc}^{\alpha \beta} \right)^2 + \left( q_{ss}^{\alpha \beta} \right)^2 + \left( q_{cs}^{\alpha \beta} \right)^2 + \left( q_{sc}^{\alpha \beta} \right)^2 \right].$$

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Using the integral representation of $\delta$ functions such as
\[
\int \frac{1}{2\pi} d\phi_0^\alpha d\phi_0^\beta e^{i\phi_0^\alpha (\phi_0^\beta - \frac{i}{\hbar} \sum \cos \phi_i \cos \phi_i')} = 1,
\]
and rescaling variables as $\hat{q}_{cc}^\alpha \rightarrow N_{cc} q_{cc}^\alpha$, $\hat{Q}_{cc}^\alpha \rightarrow N \hat{Q}_{cc}^\alpha$, etc., we obtain
\[
\tilde{f} = - \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} (\beta N n)^{-1} \log \left\{ \int \sum_{\beta} N_{cc} e^{\int N \Phi (\Phi + \Psi)} \right\}, \tag{A4}
\]
\[
\Phi = i \sum_{\alpha} \left[ \hat{Q}_{cc}^\alpha \hat{Q}_{cc}^\alpha + \hat{Q}_{cs}^\alpha \hat{Q}_{cs}^\alpha + \hat{Q}_{ss}^\alpha \hat{Q}_{ss}^\alpha \right] + i \sum_{\alpha < \beta} \left[ \hat{q}_{cc}^\alpha \hat{q}_{cc}^\beta + \hat{q}_{cs}^\alpha \hat{q}_{cs}^\beta + \hat{q}_{ss}^\alpha \hat{q}_{ss}^\beta + \hat{q}_{sc}^\alpha \hat{q}_{sc}^\beta \right] + \frac{\beta^2 J^2}{4} \sum_{\alpha} \left[ \left( Q_{cc}^\alpha \right)^2 + \left( Q_{cs}^\alpha \right)^2 + 2 \left( Q_{ss}^\alpha \right)^2 \right] + 2 \sum_{\alpha < \beta} \left[ \left( q_{cc}^\alpha \right)^2 + \left( q_{cs}^\alpha \right)^2 + \left( q_{ss}^\alpha \right)^2 \right],
\]
\[
\Psi = \frac{1}{N} \log \left[ \int \prod_{\alpha} d\phi^\alpha \right] \exp \left[ - \frac{1}{2} \sum_{\alpha} \left( \hat{Q}_{cc}^\alpha \cos^2 \phi_i^\alpha + \hat{Q}_{ss}^\alpha \sin^2 \phi_i^\alpha + \hat{Q}_{cs}^\alpha \cos \phi_i^\alpha \sin \phi_i^\alpha \right) \right] \tag{A5}
\]
where $d\mathbf{q} = \prod_{\alpha, \beta} \left( \frac{N_{cc}^\alpha dQ_{cc}^\alpha}{2\pi} \frac{N_{cs}^\alpha dQ_{cs}^\alpha}{2\pi} \frac{N_{ss}^\alpha dQ_{ss}^\alpha}{2\pi} \frac{N_{sc}^\alpha dQ_{sc}^\alpha}{2\pi} \right) \prod_{\alpha < \beta} \left( \frac{N_{cc}^\alpha dq_{cc}^\alpha}{2\pi} \frac{N_{cs}^\alpha dq_{cs}^\alpha}{2\pi} \frac{N_{ss}^\alpha dq_{ss}^\alpha}{2\pi} \frac{N_{sc}^\alpha dq_{sc}^\alpha}{2\pi} \right)$. Since we consider $N \rightarrow \infty$, the integration is estimated by the saddle point of $\Phi + \Psi$:
\[
\tilde{f} = - \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} (\beta N n)^{-1} \log \left\{ \int d\mathbf{q} e^{N \Phi (\Phi + \Psi)} \right\} \sim - \lim_{n \rightarrow 0} (\beta n)^{-1} \exp (\Phi + \Psi). \tag{A6}
\]
Now, let us consider the replica symmetric solution:
\[
Q_{cc}^\alpha = Q_{cc}, \quad Q_{cs}^\alpha = Q_{cs}, \quad Q_{ss}^\alpha = Q_{ss}, \quad Q_{sc}^\alpha = Q_{sc}, \quad Q_{cc}^\alpha = Q_{cc}, \quad Q_{ss}^\alpha = Q_{ss}, \tag{A7}
\]
\[
Q_{cc}^\alpha = q_{cc}, \quad Q_{cs}^\alpha = q_{cs}, \quad Q_{ss}^\alpha = q_{ss}, \quad Q_{sc}^\alpha = q_{sc}.
\]
Then, by changing conjugate variables from $\hat{q}_{cc} \rightarrow i\hat{q}_{cc}, \hat{Q}_{cc} \rightarrow i\hat{Q}_{cc},$ etc., we obtain
\[
\lim_{n \rightarrow 0} \frac{1}{n} \Phi_{RS} = - (\hat{Q}_{cc} Q_{cc} + \hat{Q}_{cs} Q_{cs} + \hat{Q}_{ss} Q_{ss}) + \frac{1}{2} (\hat{q}_{cc} q_{cc} + \hat{q}_{cs} q_{cs} + \hat{q}_{ss} q_{ss} + \hat{q}_{sc} q_{sc})
\]
\[
+ \frac{\beta^2 J^2}{4} \left( Q_{cc}^2 + Q_{cs}^2 + 2 Q_{ss}^2 - q_{cc}^2 - q_{cs}^2 - q_{sc}^2 \right),
\]
\[
\lim_{n \rightarrow 0} \frac{1}{n} \Psi_{RS} = \lim_{n \rightarrow 0} \frac{1}{n} \log \left( \int \prod_{\alpha} d\phi^\alpha e^{\int L} \right), \tag{A10}
\]
\[
L = \sum_{\alpha} \left( \hat{Q}_{cc}^\alpha \cos^2 \phi^\alpha + \hat{Q}_{ss}^\alpha \sin^2 \phi^\alpha + \hat{Q}_{cs}^\alpha \cos \phi^\alpha \sin \phi^\alpha \right)
\]
\[
+ \sum_{\alpha < \beta} \left( q_{cc}^\alpha \cos \phi^\beta \cos \phi^\alpha + q_{ss}^\alpha \sin \phi^\beta \sin \phi^\alpha + q_{cs}^\alpha \cos \phi^\alpha \sin \phi^\beta + q_{sc}^\alpha \sin \phi^\alpha \cos \phi^\beta \right). \tag{A11}
\]
By using the Hubbard-Stratonovich transformation, $e^L$ is rewritten as
\[
e^L = \exp \left[ \left( \hat{Q}_{cc} - \frac{1}{2} \hat{q}_{cc} \right) \sum_{\alpha} \cos^2 \phi^\alpha + \left( \hat{Q}_{ss} - \frac{1}{2} \hat{q}_{ss} \right) \sum_{\alpha} \sin^2 \phi^\alpha + \left( \hat{Q}_{cs} - \hat{q}_{cs} + \hat{q}_{sc} \right) \sum_{\alpha} \sin \phi^\alpha \cos \phi^\alpha \right]
\]
\[
\times \int Dx \int Dy \exp \left[ \sqrt{\frac{\hat{q}_{cc} q_{cc} - \hat{q}_{ss} q_{ss}}{q_{ss}}} \sum_{\alpha} \cos \phi^\alpha x + \left( \frac{q_{cc} + q_{sc}}{2 \sqrt{q_{ss}}} \right) \sum_{\alpha} \cos \phi^\alpha y \right] \tag{A12}
\]
where $Dx = (2\pi)^{-1/2} e^{-x^2/2} dx$. Then, we obtain
\[
\lim_{n \rightarrow 0} \frac{1}{n} \Psi_{RS} = \int Dx \int Dy \log \int d\phi \exp \left[ \left( \hat{Q}_{cc} - \frac{1}{2} \hat{q}_{cc} \right) \cos^2 \phi + \left( \hat{Q}_{ss} - \frac{1}{2} \hat{q}_{ss} \right) \sin^2 \phi + \left( \hat{Q}_{cs} - \hat{q}_{cs} + \hat{q}_{sc} \right) \sin \phi \cos \phi \right.
\]
\[
+ \sqrt{\frac{q_{cc} q_{ss} - \hat{q}_{ss}^2}{q_{ss}}} \cos \phi x + \left( \frac{q_{cc} + q_{sc}}{2 \sqrt{q_{ss}}} \right) \cos \phi + \sqrt{\frac{q_{cc} q_{ss}}{q_{ss}}} \sin \phi \right]. \tag{A13}
\]
\( \tilde{f}_{RS} \) is expressed as
\[
\tilde{f}_{RS} = -\frac{1}{\beta} \lim_{\beta \to 0} \frac{1}{\beta} (\Phi_{RS} + \Psi_{RS})
\]
\[
= -\frac{1}{\beta} \left\{ -\left( \frac{\beta^2 J^2}{4} q_{cc}^2 + q_{ss}^2 + q_{cs}^2 - Q_{cc}^2 - Q_{ss}^2 - 2Q_{cs}^2 \right) + \int Dx \int D y \log \int d \phi \exp \left[ \left( \frac{\beta^2 J^2}{2} Q_{ss} \right) \cos^2 \phi + \left( \frac{\beta^2 J^2}{2} Q_{cc} \right) \sin^2 \phi + \left( \frac{\beta^2 J^2}{2} Q_{cs} \right) \sin \phi \cos \phi \right] \right\},
\]
\[
M(\phi|x, y) = \exp \left[ \frac{\beta^2 J^2}{2} (Q_{cc} - q_{cc}) \cos^2 \phi + \frac{\beta^2 J^2}{2} (Q_{ss} - q_{ss}) \sin^2 \phi + \frac{\beta^2 J^2}{2} (Q_{cs} - q_{cs}) \sin \phi \cos \phi \right]
\]
\[
+ \beta J \left( \frac{q_{cc} q_{ss} - \sqrt{q_{cc} q_{ss}^3}}{q_{ss}} \cos \phi \sin \phi + \sqrt{q_{ss}} \sin \phi \right),
\]
From this, we obtain the following SPPs:
\[
Q_{cc} = [\cos^2 \phi], \quad Q_{ss} = [\sin^2 \phi] = 1 - Q_{cc}, \quad Q_{cs} = [\sin \phi \cos \phi],
\]
\[
q_{cc} = [\cos \phi]^2, \quad q_{ss} = [\sin \phi]^2, \quad q_{cs} = [\sin \phi \cos \phi] = q_{sc},
\]
Using the above relations, \( f_{RS} \) and \( M(\phi|x, y) \) are now expressed as
\[
\tilde{f}_{RS} = -\frac{\beta J^2}{4} \left( q_{cc}^2 + q_{ss}^2 + 2q_{cs}^2 - 1 + 2Q_{cc}(1 - Q_{cc}) - 2Q_{cs}^2 \right) - \frac{1}{\beta} \int Dx \int D y \log \int d \phi M(\phi|x, y),
\]
\[
M(\phi|x, y) = \exp \left[ \frac{\beta^2 J^2}{2} (Q_{cc} - q_{cc} \cos^2 \phi + \frac{\beta^2 J^2}{2} (Q_{ss} - q_{ss}) \sin^2 \phi + \frac{\beta^2 J^2}{2} (Q_{cs} - q_{cs}) \sin \phi \cos \phi \right]
\]
\[
+ \beta J \left( \frac{q_{cc} q_{ss} - \sqrt{q_{cc} q_{ss}^3}}{q_{ss}} \cos \phi \sin \phi + \sqrt{q_{ss}} \sin \phi \right),
\]
As discussed in the main text, \( Q_{cc} = Q_{ss} = \frac{1}{2} \); \( Q_{cs} = 0 \), \( q_{cc} = q_{ss} \), and \( q_{cs} = 0 \) follow. Thus, we obtain
\[
\tilde{f}_{RS} = -\frac{\beta J^2}{2} q_{cc}^2 - \frac{1}{\beta} \int Dx \int D y \log \int d \phi M(\phi|x, y),
\]
\[
M(\phi|x, y) = \exp \left[ -\frac{\beta J^2}{2} q_{cc} + \beta J \sqrt{q_{cc}} \cos \phi \sin \phi \right],
\]
where we omit irrelevant constants. Now, we introduce the polar coordinates, \( x = r \cos \theta \), \( y = r \sin \theta \). Then, we have
\[
\tilde{f}_{RS} = -\frac{\beta J^2}{2} q_{cc}^2 = -\frac{1}{\beta} \int_0^\infty \int_0^{2\pi} d r e^{-\frac{\beta J^2}{2} r^2} \int_0^{2\pi} d \theta \log \int d \phi M(\phi|r, \theta),
\]
\[
M(\phi|r, \theta) = e^{-\frac{\beta J^2}{2} q_{cc} + \beta J \sqrt{q_{cc}} r \cos \phi r \sin \phi},
\]
By performing integration, we have

\[
\bar{f}_{RS} = -\frac{\beta J^2}{2} q_{cc}^2 - \frac{1}{\beta} \int_0^\infty dr e^{-\frac{1}{2}r^2} r \log \left[ 2\pi I_0(\beta J \sqrt{q_{cc}}r) e^{-\frac{\beta J^2}{4} q_{cc}^2} \right]
\]

\[
= -\frac{\beta J^2}{2} q_{cc}^2 + \frac{\beta J^2}{2} q_{cc}^2 - \frac{1}{\beta} \int_0^\infty dr e^{-\frac{1}{2}r^2} r \log \left[ 2\pi I_0(\beta J \sqrt{q_{cc}}r) \right].
\] (A27)

\( I_n(x) \) is the \( n \)th order modified Bessel function of the first kind. The SPE becomes

\[
-\beta J^2 q_{cc} + \frac{\beta J^2}{2} q_{cc}^2 - \frac{1}{\beta} \int_0^\infty dr e^{-\frac{1}{2}r^2} r \log \left[ \frac{I_1\left(\frac{\beta J \sqrt{q_{cc}}r}{2}\right)}{I_0\left(\frac{\beta J \sqrt{q_{cc}}r}{2}\right)} \right] \beta J e^{-\frac{\beta J^2}{4} q_{cc}^2} = 0.
\] (A28)

Since the spin glass order parameter is \( q = 2q_{cc} \), we obtain

\[
q = 1 - k_B T \frac{\sqrt{2}}{J} \int_0^\infty dr \frac{r^2 e^{-\frac{1}{2}r^2}}{I_0\left(\frac{\beta J \sqrt{q_{cc}}r}{2}\right)} \frac{I_1\left(\frac{\beta J \sqrt{q_{cc}}r}{2}\right)}{I_0\left(\frac{\beta J \sqrt{q_{cc}}r}{2}\right)}.
\] (A29)

This is nothing but the equation for \( q \) derived by Sherrington and Kirkpatrick [10].