Statistical Mechanical Analysis of CDMA Multiuser Detectors

—— *AT Stability and Entropy of the RS Solution, and 1RSB Solution* ——

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A class of code-division multiple-access (CDMA) multiuser detectors in the large-system limit has been investigated by Tanaka by means of statistical mechanics, i.e. the replica method. He evaluated analytically the bit-error rate, the multiuser efficiency, etc. of Marginal-Posterior-Mode (MPM) detectors under the replica symmetry (RS) ansatz. In this paper, to investigate the breaking of the replica symmetry, we study the de Almeida-Thouless (AT) stability, the entropy, and the free energy of the RS solutions for MPM detectors including individually optimum (IO) ones.

§1. Introduction

Tanaka[1)] studied a class of code-division multiple-access (CDMA) multiuser detectors in the large-system limit by using a statistical mechanical method. First, we give a brief introduction of the model. See Ref. 1) for details.

We consider the N-user CDMA channel

$$r^\mu = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} s_k^\mu x_k + n^\mu.$$

Here, $x_k \in \{1, -1\}$ is the information bit of user k, $\{s_k^\mu; \mu = 1, \cdots, p\}$ is the spreading sequence of user k, n^μ is the white gaussian noise with the mean 0 and the variance σ_0^2. Further, we define $\boldsymbol{r} \equiv (r^1, \cdots, r^p)^T$, $\boldsymbol{x} \equiv (x_1, \cdots, x_N)^T$, $\boldsymbol{n} \equiv (n^1, \cdots, n^p)^T$, $(S)_{ij} \equiv s_j^i, i = 1, \cdots, p, j = 1, \cdots, N$. Then, $\boldsymbol{r} = \frac{1}{\sqrt{N}} S\boldsymbol{x} + \boldsymbol{n}$. By the Bayesian formula, the posterior distribution is given by

$$p(\boldsymbol{x}|\boldsymbol{r}, S) = [Z(\boldsymbol{r}, S)]^{-1} \exp\Big(-\frac{1}{2\sigma^2} ||\boldsymbol{r} - N^{-1/2} S\boldsymbol{x}||^2 \Big), \tag{1·1}$$

$$Z(\boldsymbol{r}, S) = \sum_{\boldsymbol{x}} \exp\Big(-\frac{1}{2\sigma^2} ||\boldsymbol{r} - N^{-1/2} S\boldsymbol{x}||^2 \Big), \tag{1·2}$$

where σ^2 is the variance of the posterior gaussian distribution. To evaluate Z, using the replica method and introducing n replicated random variables $\boldsymbol{x}_a = \{x_{a1}, \cdots, x_{aN}\}$, $a = 1, \cdots, n$, Tanaka obtained the saddle point equations (SPEs) for macroscopic parameters $\{R, q, \hat{R}, \hat{q}\}$. $\boldsymbol{x}_0$ being the true information-bit vector, R and q are defined as $R = \frac{1}{N} \sum_{k=1}^{N} x_{0k} x_{ak}$ for $a \geq 1$ and $q = \frac{1}{N} \sum_{k=1}^{N} x_{ak} x_{bk}$ for $a \neq b, a, b \geq 1$

under the replica symmetry (RS) ansatz. $\hat{R}$ and $\hat{q}$ are conjugate to R and q, respectively. Then, the SPEs are

$$\hat{q} = \frac{\alpha\beta^2(q - 2R + 1 + \beta_0^{-1})}{[1 + \beta(1-q)]^2}, \tag{1·3}$$

$$\hat{R} = \frac{\alpha\beta}{1 + \beta(1-q)}, \tag{1·4}$$

$$R = \int Dz \tanh(\sqrt{\hat{q}}z + \hat{R}), \tag{1·5}$$

$$q = \int Dz \tanh^2(\sqrt{\hat{q}}z + \hat{R}), \tag{1·6}$$

where $\alpha \equiv \frac{p}{N}$, $\beta \equiv \frac{1}{\sigma^2}$, $\beta_0 \equiv \frac{1}{\sigma_0^2}$ and $Dz \equiv \frac{dz}{\sqrt{2\pi}} e^{-z^2/2}$. For later use, we define "temperatures" as $T_0 = \beta_0^{-1}$ and $T = \beta^{-1}$.

In this model, Tanaka studied the bit-error rate, the multiuser efficiency, etc. of the individually optimum (IO) and jointly optimum (JO) multiuser detectors.

In this paper, to investigate the breaking of the replica symmetry, we study the AT stability and the zero-entropy conditions for the RS solutions.

In the next section, we give the SPEs for the 1-step replica symmetry breaking (1RSB) solutions and numerical results of the AT-line and zero-entropy line. In §3, discussion is given.

§2. AT-line, zero-entropy line, and 1RSB solution

The entropy and the eigenvalue of the replicon mode λ are expressed as

$$\begin{aligned} s_{RS} &= -\frac{\partial}{\partial T} f_{RS} \\ &= \frac{\hat{R}(1-q-2R)}{2} - \hat{q}(1-q) - \frac{\alpha}{2}\ln[1+\beta(1-q)] + \int Dz \ln[2\cosh(\sqrt{\hat{q}}z + \hat{R})], \\ \lambda &= \frac{\alpha\beta^2}{[1+\beta(1-q)]^2} \int Dz\, \mathrm{sech}^4(\sqrt{\hat{q}}z + \hat{R}) - 1. \end{aligned}$$

We calculated the entropy and the eigenvalue λ at the RS solutions for several ranges of the parameters α, β_0 and β. We found that there are two types of anomalies implying occurrence of RSB: One is destabilization of the replicon mode, which is signalled by positive λ. Another is freezing of the system, which is characterized by negative entropy. For given pairs of α and β_0, neither types of anomaly occur at small enough β. As β becomes larger, either the destabilization or the freezing or both may occur, depending on the parameters α and β_0. When the destabilization occurs first, we call it case 1. Assuming the 1RSB we obtain the following SPEs:

$$q_1 = \int Dz \frac{1}{K} \int Dt\, \Omega^m \tanh^2(\sqrt{\hat{q}_0}z + \sqrt{\hat{q}_1 - \hat{q}_0}t + \hat{R}), \tag{2·1}$$

$$q_0 = \int Dz \left\{ \frac{1}{K} \int Dt\, \Omega^m \tanh(\sqrt{\hat{q}_0}z + \sqrt{\hat{q}_1 - \hat{q}_0}t + \hat{R}) \right\}^2, \tag{2·2}$$

$$R = \int Dz \frac{1}{K} \int Dt \Omega^m \tanh(\sqrt{\hat{q}_0} z + \sqrt{\hat{q}_1 - \hat{q}_0} t + \hat{R}), \tag{2·3}$$

$$\hat{q}_1 = \alpha\beta^2 \frac{q_1 - q_0}{(1+\beta(1-q_1))[1+\beta(1-q_1)+m\beta(q_1-q_0)]} + \hat{q}_0, \tag{2·4}$$

$$\hat{q}_0 = \frac{\alpha\beta^2(1+\beta_0^{-1}+q_0-2R)}{[1+\beta(1-q_1)+m\beta(q_1-q_0)]^2}, \tag{2·5}$$

$$\hat{R} = \frac{\alpha\beta}{1+\beta(1-q_1)+m\beta(q_1-q_0)}, \tag{2·6}$$

$$\frac{(\hat{q}_1-\hat{q}_0)\{1+\beta(1-q_1)+m\beta q_1\}}{2\beta} = \frac{\alpha}{2m} \ln\left[\frac{1+\beta(1-q_1)+m\beta(q_1-q_0)}{1+\beta(1-q_1)}\right]$$
$$-\frac{1}{m} I + \int Dz \frac{1}{K} \int Dt \Omega^m \ln \Omega, \tag{2·7}$$
$$I = \int Dz \ln K, \quad K = \int Dt \Omega^m, \quad \Omega = 2\cosh(\sqrt{\hat{q}_0} z + \sqrt{\hat{q}_1 - \hat{q}_0} t + \hat{R}).$$

When the freezing occurs first, we call it case 2. We take the Krauth-Mézard limit $q_1 \to 1, \hat{q}_1 \to \infty$ as in the case of Ising perceptron,[2)] and we obtain as the free energy

$$f_{1RSB}(q_1 = 1, \hat{q}_1 = \infty, q_0, \hat{q}_0, R, \hat{R}, m, \beta) = f_{RS}(q_0, m^2\hat{q}_0, R, m\hat{R}, m\beta). \tag{2·8}$$

Thus, the SPEs are those for the RS solution plus zero entropy condition, $s_{RS} = 0$. When the solution of these equations are denoted as $q_c, \hat{q}_c, R_c, \hat{R}_c$ and β_c, then, the 1RSB solution is expressed by

$$q_0 = q_c,\ \hat{q}_0 = m^{-2}\hat{q}_c,\ R = R_c,\ \hat{R} = m^{-1}\hat{R}_c,\ m = \frac{\beta_c}{\beta}.$$

Since $m \le 1$, the 1RSB solution is valid for $T \le T_c$.

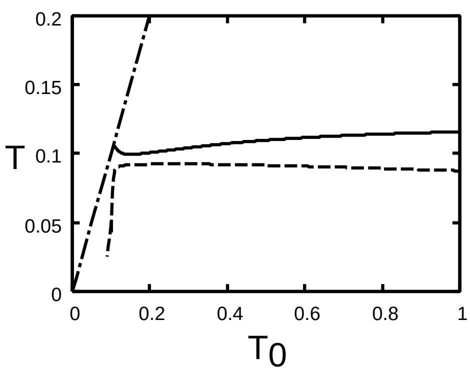

Fig. 1. Case 1. $\alpha = 0.7$. AT-line(solid curve) and the zero-entropy line(dashed curve) in space of (T_0, T).

Now, let us explain the numerical results in detail. In space of T_0 and T with fixed α, we calculated the AT-line ($T = T_{AT}(T_0)$) and the zero-entropy line ($T = T_s(T_0)$) for the RS solution. When α is small, the AT-line is located below the zero-entropy line. However, when α is large, the AT-line is above the zero-entropy line. We show two examples of the results in Figs. 1 and 2. When $\alpha = 0.7$ (Fig. 1), case 1 occurs as T is decreased with T_0 fixed. On the other hand, when $\alpha = 0.4$ (Fig. 2), since the zero-entropy line is above the AT-line, case 2 occurs as T is decreased with fixed T_0. Now, let us focus on case 2. As is seen from Fig. 2, the zero-entropy line intersects with the $T_0 = T$ line, the so-called Nishimori-line, which is the case of IO multiuser detectors. This implies that the

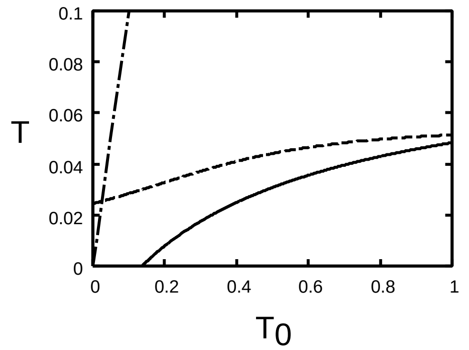

Fig. 2. Case 2. $\alpha = 0.4$. AT-line(solid curve) and the zero-entropy line(dashed curve) in space of (T_0, T).

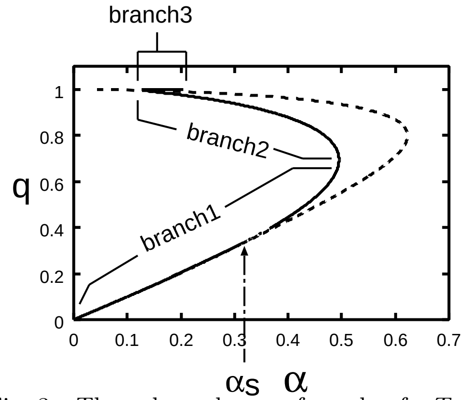

Fig. 3. The α dependences of q and q_0 for $T_0 = T = 0.01$. Solid curve: q, the RS solution. Dashed curve: q_0, the 1RSB solution.

breaking of the replica symmetry takes place on the Nishimori-line. However, from the gauge-theory-based consideration by Nishimori,[3] any RSB should not take place on the Nishimori-line for thermodynamically dominant states.

In search of an explanation for this disagreement, we studied further details of case 2. We fixed $T_0 = T = 0.01$ and varied α. Then, there are three branches of the RS solutions. See Fig. 3. Let α_s be the value of α where the s_{RS} of branch 1 becomes 0 and let f_i be the free energy of branch i, $i = 1, 2, 3$. We found that at some value of α, say α_f, f_1 and f_3 coincide. Further we found $f_1 < f_3$ for $\alpha < \alpha_f$ and $f_1 > f_3$ for $\alpha > \alpha_f$. Thus at $\alpha = \alpha_f$, the thermodynamic transition from branch 1 to branch 3 takes place when α increases below α_f. Further, we found $\alpha_s \simeq \alpha_f$. Thus, at nearly the same value of α where the solution corresponding to the branch 1 experiences the 1RSB, it becomes metastable.

§3. Discussion

In this paper, we studied two types of the replica symmetry breaking. In particular, we found that there exists a 1RSB solution on the Nishimori-line. Since the 1RSB solution appears from the metastable state, this result is consistent with the above-mentioned result by Nishimori.

Let us discuss the bit-error rate. In case 2, the RS solutions of branch 1 have rather high bit-error rates compared to those in branch 3. As α increases, a discrete transition from the state with the higher bit-error rate to the state with the lower one takes place. On the other hand, in case 1, there is only one branch, along which q and R monotonically increase, and the bit-error rate decreases, as α increases.

As for the bit-error rate in the 1RSB solution and the details of spinodal lines, we have not obtained definite results yet. These are future problems.

References

1) T. Tanaka, IEEE Trans. Inform. Theory **48** (2002), 2888.
2) W. Krauth and M. Mézard, J. de Phys. **50** (1989), 3057.
3) H. Nishimori, Europhys. Lett. **57** (2002), 302.