Correspondence between Phase Oscillator Network and Classical XY Model with the Same Infinite-Range Interaction in Statics

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We study phase oscillator networks with distributed natural frequencies and classical XY models, both of which have a class of infinite-range interactions in common. We find that the integral kernel of the self-consistent equations (SCEs) for oscillator networks corresponds to that of the saddle point equations (SPEs) for XY models, and that the quenched randomness (distributed natural frequencies) corresponds to thermal noise. We find a sufficient condition that the probability density of natural frequency distributions is one-humped, so that the kernel in an oscillator network is strictly decreasing, as in the XY model. Furthermore, taking the uniform and Mexican-hat-type interactions, we prove the one-to-one correspondence between the solutions of the SCEs and SPEs. As an application of the correspondence, we study the associative-memory-type interaction. In the XY model with this interaction, there exists a peculiar one-parameter family of solutions. For the oscillator network, we find a nontrivial solution, i.e., a limit cycle oscillation.

Synchronization phenomena prevail in nature[1,2] and have drawn many researchers. Among them, Winfree studied biological rhythms and introduced a phase description.[3] Later, Kuramoto proposed a seminal model of synchronization-desynchronization transitions, the so-called Kuramoto model.[4] Since then, many studies on the Kuramoto

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model and its extensions have been carried out.[5,6] On the other hand, the classical XY models have been studied intensively and extensively, mainly for short range interactions.[7] If the interactions are the same, complex order parameters are also the same in both models. In the course of our study of a phase oscillator network with an infinite-range interaction,[8] we investigated the classical XY model with the same interaction and found that complex order parameters obey similar equations in both models.[9]

It is obvious that the phase oscillator network with the uniform natural frequency is equivalent to the classical XY model with a temperature of zero if the interaction is common in the two models. It is quite nontrivial, however, whether there exists some correspondence between the phase oscillator network with distributed natural frequencies and the classical XY model at nonzero temperatures, because the former is a deterministic nonequilibrium system with quenched randomness, whereas the latter is a system subject to thermal noise, which changes with time, and is analyzed on the basis of the equilibrium statistical mechanics. In this paper, we report that there exists some correspondence between these two quite different models. We treat a class of interactions for which the Hamiltonian is expressed by order parameters, and derive the correspondence between the probability density functions for phases, and between the self-consistent equations (SCEs) for the phase oscillator network and the saddle point equations (SPEs) for the XY model. We also find a sufficient condition for the probability density of natural frequency distributions so that the precise correspondence holds in both models, and we also find that the quenched randomness corresponds to thermal noise. Furthermore, we study the uniform interaction and the Mexican-hat-type interaction on a circle, and prove the one-to-one correspondence of the solutions in both models. Finally, as an application, we study the associative-memory-type interaction. For the XY model with this interaction, there exists a peculiar solution, namely, a one-parameter family of solutions that we call the continuous attractor.[10] Through the correspondence, we immediately obtain the SCEs for the oscillator network. We theoretically and numerically study both models and find that the continuous attractor changes to a noisy limit cycle oscillation in the oscillator network.

<u>Phase oscillator network</u>

Let us consider N phase oscillators. Let ϕ'_j be the phase of the jth oscillator, and assume that it obeys the following differential equation:

$$\frac{d}{dt}\phi'_j \quad = \quad \omega_j + \sum_{k=1}^{N} J_{jk} \sin(\phi'_k - \phi'_j). \tag{1}$$

Here, ω_j is the natural frequency and it is drawn from the probability density $g(\omega)$. We assume that $J_{jk} = J_{kj}$, the mean value of ω is ω_0, and $g(\omega)$ is symmetric with respect to ω_0,

$$g(\omega_0 + x) \quad = \quad g(\omega_0 - x). \tag{2}$$

We put $\phi_j = \phi'_j - \omega_0$ and define A_j and α_j by

$$A_j e^{i\alpha_j} \quad = \quad \sum_{k=1}^{N} J_{jk} e^{i\phi_k}. \tag{3}$$

Since we are interested in stationary states, we assume that A_j and α_j do not depend on time. By defining $\psi_j = \phi_j - \alpha_j$, the evolution equation becomes

$$d\psi_j/dt \quad = \quad \omega_j - \omega_0 - A_j \sin\psi_j. \tag{4}$$

Let $\hat{n}(\phi', t, j)$ be the probability density of ϕ' for the jth oscillator at time t. Assuming a stationary rotation of the probability density and defining $n(\psi, j) \equiv \hat{n}(\phi', t, j)$, the continuity equation becomes

$$\frac{\partial}{\partial t} n(\psi, j) \quad = \quad -\frac{\partial}{\partial \psi}\Big(\big(\omega_j - \omega_0 - A_j \sin\psi\big) n(\psi, j) \Big). \tag{5}$$

Its stationary solution is

$$\big(\omega_j - \omega_0 - A_j \sin\psi_j\big) n(\psi, j) \quad = \quad C_j, \tag{6}$$

$$n(\psi, j) \quad = \quad n_s(\psi, j) + n_{ds}(\psi, j), \tag{7}$$

where n_s and n_{ds} are densities for the synchronized and desynchronized oscillators, respectively. For the stable synchronized oscillators, we obtain

$$n_s(\psi, j) \quad = \quad g(\omega_0 + A_j \sin\psi) A_j \cos\psi, \ \ |\psi| < \pi/2. \tag{8}$$

<u>XY model</u>

The classical XY spins are denoted by $\boldsymbol{X}_j = (\cos\phi_j, \sin\phi_j), j = 1, 2, \cdots, N$. The Hamiltonian is given by

$$H \quad = \quad -\sum_{j<k} J_{jk} \cos(\phi_j - \phi_k) = -\frac{1}{2} \sum_{j,k} J_{jk} \cos(\phi_j - \phi_k) + C, \tag{9}$$

where $C = \sum_j J_{jj}/2$. The equilibrium state is described by the canonical distribution $P_{eq} = e^{-\beta H}/Z$, where Z is the partition function, $\beta = 1/T$, and T is the temperature.

We put $k_B = 1$.

Interaction and order parameters

We consider the following interaction:

$$J_{jk} = \frac{1}{N}\sum_{l=1}^{L} a_l q_{l,j} q_{l,k}, \tag{10}$$

where $a_l > 0$ and $q_{l,j}$ are real numbers. We define the order parameters as

$$Q_l e^{i\Phi_l} = Q_{l,\mathrm{R}} + iQ_{l,\mathrm{I}} = \frac{1}{N}\sum_{j=1}^{N} q_{l,j} e^{i\phi_j},\ l = 1, \cdots, L. \tag{11}$$

Therefore, in the XY model, the Hamiltonian is expressed as

$$H = -\frac{N}{2}\sum_{l=1}^{L} a_l (Q_{l,\mathrm{R}}^2 + Q_{l,\mathrm{I}}^2) + C. \tag{12}$$

By the saddle point method,[11] we obtain the partition function and the probability density function $n(\phi, j)$ of ϕ_j for the jth spin as

$$Z \propto \exp[N(-\frac{\beta}{2}\sum_{l=1}^{L} a_l Q_l^2 + \frac{1}{N}\sum_{j=1}^{N}\Omega_j)], \tag{13}$$

$$\exp[\Omega_j] = \int_0^{2\pi} d\phi_j \exp[\beta \sum_{l=1}^{L} a_l q_{l,j}(Q_{l,\mathrm{R}}\cos\phi_j + Q_{l,\mathrm{I}}\sin\phi_j)]$$

$$= 2\pi I_0(\beta \Xi_j), \tag{14}$$

$$\Xi_j = \sqrt{(\sum_{l=1}^{L} a_l q_{l,j} Q_{l,\mathrm{R}})^2 + (\sum_{l=1}^{L} a_l q_{l,j} Q_{l,\mathrm{I}})^2}, \tag{15}$$

$$\Xi_j \cos\phi_j^0 = \sum_{l=1}^{L} a_l q_{l,j} Q_{l,\mathrm{R}},\ \Xi_j \sin\phi_j^0 = \sum_{l=1}^{L} a_l q_{l,j} Q_{l,\mathrm{I}}, \tag{16}$$

$$n(\phi, j) = \frac{\exp[\beta \Xi_j \cos(\phi - \phi_j^0)]}{2\pi I_0(\beta \Xi_j)}. \tag{17}$$

Here, $I_n(x)$ is the nth-order modified Bessel function,

$$I_n(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp[x\cos\theta]\cos(n\theta). \tag{18}$$

$n(\phi, j)$ is the so-called von Mises distribution. This function corresponds to Eq. (8). The SPEs are

$$Q_l e^{i\Phi_l} = \frac{1}{N}\sum_{j=1}^{N}\int_0^{2\pi} d\phi_j \exp[-\Omega_j + \beta\sum_{l'=1}^{L} q_{l',j}(Q_{l',\mathrm{R}}\cos\phi_j + Q_{l',\mathrm{I}}\sin\phi_j)] q_{l,j} e^{i\phi_j}$$

$$= \frac{1}{N}\sum_{j=1}^{N}\frac{I_1(\beta\Xi_j)}{I_0(\beta\Xi_j)}q_{l,j}e^{i\phi_j^0}. \quad (19)$$

Furthermore, we obtain the following relation from Eqs. (3) and (10):

$$A_j e^{i\alpha_j} = \Xi_j e^{i\phi_j^0}. \quad (20)$$

In the oscillator network, the SCEs are

$$Q_l e^{i\Phi_l} = \frac{1}{N}\sum_{j=1}^{N} 2\int_0^{\pi/2} d\psi g(\omega_0 + A_j\sin\psi)A_j\cos^2\psi\ q_{l,j}e^{i\alpha_j}. \quad (21)$$

The desynchronized solutions do not contribute to the order parameters due to assumption (2).

Correspondence of integral kernels and that of randomness

Let us define the following functions and coefficients:

$$u(x) \equiv I_1(x)/[xI_0(x)], \quad (22)$$

$$\bar{g}_{\omega_0,\sigma}(x) \equiv 2\int_0^{\pi/2} d\psi g(\omega_0 + x\sin\psi)\cos^2\psi, \quad (23)$$

$$\bar{u}_\beta(x) \equiv \beta u(\beta x). \quad (24)$$

Using these functions and Eq. (20), SPEs (19) and SCEs (21) are rewritten as

$$Q_l e^{i\Phi_l} = \frac{1}{N}\sum_{j=1}^{N} A_j\bar{u}_\beta(A_j)q_{l,j}e^{i\alpha_j}, \quad (25)$$

$$Q_l e^{i\Phi_l} = \frac{1}{N}\sum_{j=1}^{N} A_j\bar{g}_{\omega_0,\sigma}(A_j)q_{l,j}e^{i\alpha_j}. \quad (26)$$

From these equations, we find that $\bar{g}_{\omega_0,\sigma}(x)$ and $\bar{u}_\beta(x)$ correspond. If we derive the concrete equations for order parameters in one model, we immediately obtain them in the other model. We call these functions the integral kernels because these equations become integral equations in some cases, as seen later. Furthermore, from the value of the kernels at $x = 0$, we have the following correspondence:

$$T \Longleftrightarrow 1/[\pi g(\omega_0)](= \sqrt{2/\pi}\sigma), \quad (27)$$

where the expression in the parentheses is for the Gaussian distribution, and σ is the standard deviation of the natural frequency ω. The correspondence (27) is also derived by comparing the phase transition points in both models. Equation (27) implies that the temperature corresponds to the width of the distribution of the natural frequency around the center ω_0, that is, thermal noise corresponds to the quenched randomness.

Sufficient condition under which both kernels have the same property

$\bar{u}_\beta(x)$ and $\bar{g}_{\omega_0,\sigma}(x)$ take finite values at $x = 0$, and tend to 0 as x tends to ∞. In addition to these properties, $\bar{u}_\beta(x)$ has the following property:

$$\frac{d\bar{u}_\beta}{dx}(x) < 0, \text{ for } x > 0. \tag{28}$$

A sufficient condition for the property (28) is $g'(\omega) < 0$ for $\omega > \omega_0$, that is, $g(\omega)$ has a single maximum at ω_0 and is strictly decreasing for $\omega > \omega_0$. Hereafter, we assume this property for $g(\omega)$. Using these properties, we prove the correspondence of solutions in the following.

Correspondence of solutions

Uniform interaction $J_{jk} = J_0/N$

In this case, $L = 1, a_1 = J_0$, and $q_{1,j} = 1$.The order parameter is defined as

$$Re^{i\Theta} = R_\mathrm{R} + iR_\mathrm{I} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}. \tag{29}$$

The Hamiltonian is $H = -J_0N(R_\mathrm{R}^2 + R_\mathrm{I}^2)/2 + C$. For the phase oscillator network, this is the Kuramoto model. $A_j = \Xi_j = J_0R$ and $\alpha_j = \phi_j^0 = \Theta$ follow from Eq. (3). From Eq. (26), the SCE for the order parameter R is

$$R = J_0R\bar{g}_{\omega_0,\sigma}(J_0R). \tag{30}$$

On the other hand, for the XY model, from Eq. (25), we obtain the SPE as

$$R = J_0R\bar{u}_\beta(J_0R). \tag{31}$$

Let us define $v(x)$ and $\bar{J}_0$ as

$$v(x) = \begin{cases} \bar{q}_{\omega_0,\sigma}(x)/\bar{q}_{\omega_0,\sigma}(0) = 4/(\pi g(\omega_0)) & \\ \times \int_0^{\pi/2} d\psi g(\omega_0 + x\sin\psi)\cos^2\psi, & \text{Oscillator,} \\ \bar{u}_\beta(x)/\bar{u}_\beta(0) = 2u(\beta x), & \text{XY model,} \end{cases} \tag{32}$$

$$\bar{J}_0 = \begin{cases} \bar{q}_{\omega_0,\sigma}(0)J_0 = \pi g(\omega_0)J_0/2, & \text{Oscillator,} \\ \bar{u}_\beta(0)J_0 = \beta J_0/2, & \text{XY model.} \end{cases} \tag{33}$$

We put $x = J_0R$ and $\xi = 1/\bar{J}_0$. Then, SCE and SPE become

$$\xi = v(x). \tag{34}$$

Since $v(0) = 1$ and $v(x)$ decreases monotonically to 0 as x increases from 0 to infinity, Eq. (34) has the unique solution for any $\xi \in (0, 1]$. Thus, there is a one-to-one correspondence between solutions of the SCE and SPE. The critical point is $\bar{J}_0 = 1$.

Mexican-hat-type interaction

Now, let us consider the system on a circle. We study the Mexican-hat-type interaction, which is given by

$$J_{jk} = J_0/N + (J_1/N)\cos(\theta_j - \theta_k), \tag{35}$$

where θ_j is the coordinate on the circle, $\theta_j = 2\pi j/N,\ j = 0, 1, \cdots, N-1$. The order parameters other than R are defined as

$$R_{1c}e^{i\Theta_{1c}} = \frac{1}{N}\sum_{j=1}^{N}\cos\theta_j e^{i\phi_j}, \tag{36}$$

$$R_{1s}e^{i\Theta_{1s}} = \frac{1}{N}\sum_{j=1}^{N}\sin\theta_j e^{i\phi_j}. \tag{37}$$

We define $R_1 = \sqrt{R_{1c}^2 + R_{1s}^2}$. We use θ instead of j to indicate the location. There are three nontrivial solutions: the uniform (U) solution ($R > 0, R_1 = 0$), the spinning (S) solution ($R = 0, R_1 > 0$), and the pendulum (Pn) solution ($R > 0, R_1 > 0$). See Ref. 8 for details. The uniform solution is equivalent to the solution of the Kuramoto model. Now, let us study the stable spinning solution. $R_c = R_s$ follows. We define $\bar{J}_1$ as

$$\bar{J}_1 = \begin{cases} \bar{q}_{\omega_0,\sigma}(0)J_1 = \pi g(\omega_0)J_1/2, & \text{Oscillator,} \\ \bar{u}_\beta(0)J_1 = \beta J_1/2, & \text{XY model.} \end{cases} \tag{38}$$

We put $x = J_1 R_{1c}$ and $\eta = 1/\bar{J}_1$. The SCE and SPE become

$$\eta = v(x)/2. \tag{39}$$

Therefore, there exists the unique solution of Eq. (39) for any $\eta \in (0, 1/2]$. Thus, the solutions for the SCE and SPE correspond uniquely. The critical point is $\bar{J}_1 = 2$.

Next, we study the stable pendulum solution. We define $x = J_0 R$ and $y = J_1 R_1$. The SCEs and SPEs become

$$\xi = F(x,y) = \langle v[\Lambda(x,y,\theta)]\rangle, \tag{40}$$

$$\eta = G(x,y) = \langle v[\Lambda(x,y,\theta)]\cos^2\theta\rangle, \tag{41}$$

$$\langle B\rangle = \frac{2}{\pi}\int_0^{\pi/2} d\theta B, \tag{42}$$

where $\Lambda(x,y,\theta) = \sqrt{x^2 + y^2\cos^2\theta}$. We obtain

$$F(x,0) = v(x), \tag{43}$$

$$F_y(x,y) = \partial F(x,y)/\partial y < 0, \text{ for } x \geq 0,\ y > 0. \tag{44}$$

$$\lim_{y\to\infty} F(x,y) = 0 \text{ for } x \geq 0. \tag{45}$$

Thus, for fixed $x \geq 0$, $F(x, y)$ is a decreasing function of y. For $\xi \in (0, 1]$, there is the unique solution of $v(x) = \xi$. We denote it by $x_0 = v^{-1}(\xi)$. Note that $x_0 = 0 = v^{-1}(1)$. Therefore, there exists the unique solution of Eq. (40) for any $x \in [0, x_0]$.

$$y = y(\xi, x). \tag{46}$$

We have the relations $y(\xi, x_0) = 0$ and $y(1, 0) = 0$. Substituting Eq. (46) into Eq. (41), we obtain

$$\eta = G[x, y(\xi, x)]. \tag{47}$$

It is proved that $G[x, y(\xi, x)]$ is a strictly increasing function of x for $x > 0$. Since $y(\xi, x)$ exists for $0 \leq x \leq x_0$, when $G[0, y(\xi, 0)] \leq \eta \leq G[x_0, y(\xi, x_0)]$, the solution $x(\xi, \eta)$ of Eq. (47) uniquely exists. $y(\xi, 0)$ is determined by

$$\xi = F[0, y(\xi, 0)] = \langle v[y(\xi, 0)\cos\theta] \rangle. \tag{48}$$

On the other hand, because of $v(x_0) = \xi$, $G[x_0, y(\xi, x_0)]$ is given by

$$G[x_0, y(\xi, x_0)] = G(x_0, 0) = \langle v(x_0)\cos^2\theta \rangle = \xi/2. \tag{49}$$

Thus, defining $\eta_0(\xi) \equiv G[0, y(\xi, 0)]$, the solution of Eq. (47) uniquely exists for $\eta_0(\xi) \leq \eta \leq \xi/2$. The condition $\eta \leq \xi/2$ implies $J_1 \geq 2J_0$, and this is the condition that the Pn solution emerges from the U solution.[8,9] On the other hand, the condition $\eta = \eta_0(\xi)$ indicates that the stable Pn solution becomes unstable and then disappears by merging with the unstable S solution.

Application

Now, let us consider an application of the correspondence between the two models. To obtain nontrivial results, we study the following associative-memory-type interaction:

$$J_{jk} = \frac{J}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu, \tag{50}$$

where $\boldsymbol{\xi}^\mu = (\xi_1^\mu, \xi_2^\mu, ..., \xi_N^\mu)$ is the μth pattern ($\mu = 1, 2, \cdots, p$). That is, $a_\mu = J, q_{\mu,j} = \xi_j^\mu$. We assume that $p \ll N$ and ξ_i^μ take values of ± 1, and correlate with each other as follows:

$$\langle \xi_i^\mu \xi_j^\nu \rangle = \Big(a + (1-a)\delta_{\mu,\nu}\Big)\delta_{i,j}. \tag{51}$$

The XY model with this interaction has a peculiar solution, that is, there exists a one-parameter family of solutions of the SPEs.[10] We call this solution the continuous attractor. Here, we derive the SPEs of this solution. We introduce sublattices Λ_l in which the following holds:

$$(\xi_i^1, \xi_i^2, \cdots, \xi_i^p) = (\eta_l^1, \eta_l^2, \cdots, \eta_l^p) \quad \text{for } i \in \Lambda_l, \tag{52}$$

$$\eta_{l+2^{p-1}}^\mu = -\eta_l^\mu, \ l = 1, 2, \cdots, 2^{p-1}. \tag{53}$$

The number of elements in Λ_l, $|\Lambda_l|$, is $|\Lambda_l| = N/2^p$ $(l = 1, 2, \cdots, 2^p)$. Order parameters are defined as

$$R_\mu e^{i\Theta_\mu} = R_{\mu\mathrm{R}} + iR_{\mu\mathrm{I}} = \frac{1}{N}\sum_{j=1}^N \xi_j^\mu e^{i\phi_j}, \ \mu = 1, \cdots, p. \tag{54}$$

The Hamiltonian is rewritten as

$$H = -\frac{1}{2}NJ\sum_{\mu=1}^p R_\mu^2 + C. \tag{55}$$

From Eq. (19), the SPEs become

$$R_\mu e^{i\Theta_\mu} = \beta J\langle u(x_j)\sum_{\nu=1}^p (R_{\nu\mathrm{R}} + iR_{\nu\mathrm{I}})\xi_j^\nu \xi_j^\mu\rangle, \tag{56}$$

$$\Xi_j = \sqrt{(\sum_{j=1}^N R_{\mu\mathrm{R}}\xi_j^\mu)^2 + (\sum_{j=1}^N R_{\mu\mathrm{I}}\xi_j^\mu)^2}, \tag{57}$$

where Ξ_j is redefined as Ξ_j in Eq. (15) divided by J. $x_j = \beta J\Xi_j$, and $\langle\cdot\rangle$ implies the average over $\{\xi_j\}$. We define the probability P_l that $\{\xi_i^\mu\}$ take values in the lth sublattice. The SPEs are rewritten as

$$R_{\mu\mathrm{R}} = \beta J\sum_{\nu=1}^p c_{\mu\nu}R_{\nu\mathrm{R}}, \tag{58}$$

$$R_{\mu\mathrm{I}} = \beta J\sum_{\nu=1}^p c_{\mu\nu}R_{\nu\mathrm{I}}, \tag{59}$$

$$c_{\mu\nu} = 2\sum_{l=1}^{2^{p-1}} P_l u_l \eta_l^\mu \eta_l^\nu = c_{\nu\mu}, \tag{60}$$

where $u_l = u(x_l)$, $x_l = \beta J\Xi_l$, and Ξ_l is Ξ_j evaluated at the lth sublattice. By defining $R = \sqrt{\sum_{\mu=1}^p R_\mu^2}$, we obtain additional equations from Eqs. (58) and (59) as

$$R^2 = \frac{1}{2^{p-1}}\sum_{l=1}^{2^{p-1}}\left(\frac{x_l}{\beta J}\right)^2, \tag{61}$$

$$R^2 = \frac{2}{\beta J}\sum_{l=1}^{2^{p-1}} P_l u_l x_l^2. \tag{62}$$

The SPEs of the continuous attractor are

$$c_{\mu\nu} = \delta_{\mu\nu}\frac{1}{\beta J}. \tag{63}$$

Hereafter, we study the case $a = 0$ for simplicity. For $p = 2$, from Eq. (63), $u_1 = u_2 = 1/(\beta J)$, and thus $x_1 = x_2$ follows. From Eq. (61), we obtain $R = \Xi_1 = x_1/(\beta J)$. Thus, the SPE is rewritten as

$$\bar{u}_\beta(JR) \quad = \quad 1/J, \tag{64}$$

which determines R. Thus, the continuous attractor is given by

$$0 \le R_1 \le R,\ R_1^2 + R_2^2 = R^2. \tag{65}$$

This implies that any point (R_1, R_2) on the quarter of the circle connecting two points representing patterns $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^2$ is a solution. For general p, any point on the quarter of the circle connecting any two points representing patterns $\boldsymbol{\xi}^\mu$ and $\boldsymbol{\xi}^\nu$ is a continuous attractor. We performed Markov chain Monte Carlo (MCMC) simulations for $p = 2$ and 3. We show the result for $p = 2$ in Fig. 1. We note that the trajectories of R_1 and R_2 wander but R is almost constant. As seen from Fig. 1(c), theoretical and numerical results agree quite well. Next, let us study the phase oscillator network with the same

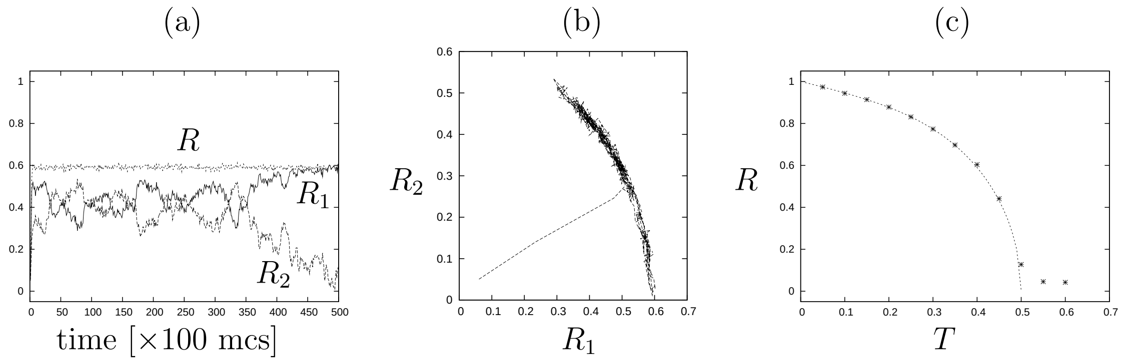

Fig. 1. XY model. $J = 1$. (a), (b) Time series of R_1, R_2, and R, and trajectories in (R_1, R_2) space obtained by Monte Carlo simulations. $N = 10^4$. $T = 0.4$. (c) T dependence of R. Curve, theoretical results; symbols, numerical results. $N = 10^4$.

interaction. The SCE is immediately obtained by the correspondence of the integral kernels,

$$\bar{q}_{\omega_0,\sigma}(JR) \quad = \quad 1/J. \tag{66}$$

This is simply the SCE of the Kuramoto model. Since we have the same relation as Eq. (65), we also obtain the continuous solution. We performed numerical integrations of Eq. (1) for $p = 2$ and 3. We took a Gaussian distribution with a mean 0 and a standard deviation σ for $g(\omega)$. We used the Euler method with the time increment $\Delta t = 0.1$. See Fig. 2. There should exist continuous stationary states, but instead, we found a noisy limit cycle oscillation. The reason for this is considered as follows. In the derivation of the SCE (21), the desynchronized oscillators do not contribute. However, in numerical simulations, the desynchronized oscillators contribute to the dynamics because N is finite. Since the continuous stationary states easily move in the marginally stable direction by perturbations, the trajectories move on the manifold of $R_1^2 + R_2^2 = R^2$. This is confirmed by Fig. 2(b). Figure 2(c) on the σ dependence of R shows fairly good agreement between the theoretical results for the continuous solution and the numerical results for the limit cycle.

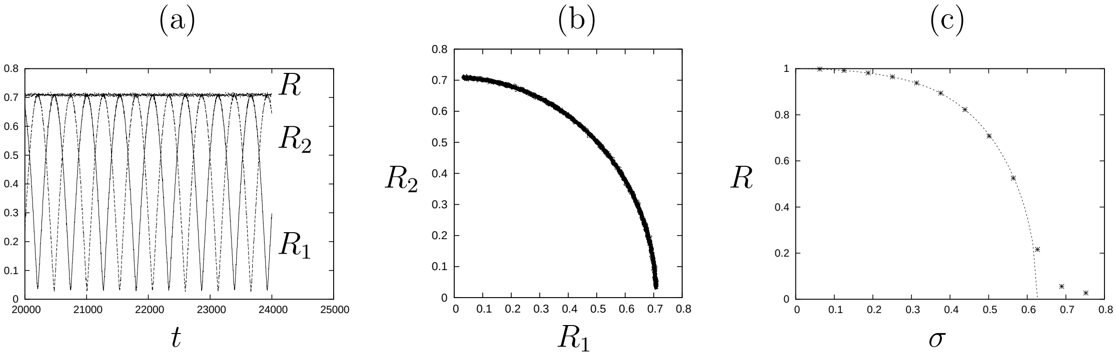

Fig. 2. Phase oscillator network. $J = 1$ (a), (b) Numerical results of time series of R_1, R_2, and R, and trajectories in (R_1, R_2) space obtained by the Euler method. $N = 10^4, \sigma = \sqrt{\frac{\pi}{2}}T$ with $T = 0.4$. (c) σ dependence of R. Curve, theoretical results; symbols, numerical results. $N = 10^4$.

In summary, we studied the correspondence between the phase oscillator networks and the classical XY models with the same infinite-range interactions. Assuming a class of interactions, we found the correspondence between the integral kernel of the SCEs for the oscillator network and that of the SPEs for the XY model. We found a sufficient condition under which the integral kernel of the SCEs for the oscillator network has the same feature as that of the SPEs for the XY model. That is, the probability density of the natural frequency distribution is one-humped. Furthermore, we found that the quenched randomness (distributed natural frequencies) corresponds to thermal noise. To study the correspondence of solutions in both models, we investigated the uniform interaction and the Mexican-hat-type interaction on a circle. We proved that the solutions uniquely correspond in both models. Upon applying this correspondence, we studied the associative-memory-type interaction, for which the XY model has a peculiar one-parameter family of solutions called the continuous attractor. We found that the continuous solution is not stable in the oscillator network, and instead a noisy limit cycle appears, which lies on the manifold where the continuous solutions exist. We consider that this is caused by the desynchronized oscillators and is a finite size effect.

When $g(\omega)$ is a uniform distribution and is therefore not one-humped, we can still prove the one-to-one correspondence of solutions for some interactions in both models. This will be reported elsewhere.

For the interactions studied in this paper, there exist several types of solution, and we found that the stabilities of the corresponding solutions in both models are the same except for the continuous solution. To study the stability of solutions in the oscillator network, we have to derive the evolution equations for order parameters and this is a very difficult problem to solve.[12)] The correspondence between the stabilities of the solutions in both models remains a future problem.

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